Math 138 Physics Based Section Assignment 7

(Q1) i) Show that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges using the comparison theorem.
ii) as i) but use the integral test.
iii) Show that the sum

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.
iv) Rewrite the sum in part iii) using difference of squares: $n^2 - 1 = (n+1)(n-1)$ and partial fractions. This way you produce two different series. Show that both of these series diverge and explain why this does not contradict what you found in part iii).
v) Show that the sum

$$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

converges. Could the ratio test make your life easier?
vi) Does the sum

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$$

converge. Does it converge absolutely?
vii) Does the sum

$$\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$$

converge. Does it converge absolutely?
(Q2) Many of the formal statements of convergence theorems are very restrictive in their assumptions. In practice, one can often use the tests in situations that don't quite meet the hypotheses as stated, BUT this must be done carefully. Consider the series

$$\sum_{n=1}^{\infty} n \exp(-n/5)$$

i) We wish to use the integral test with the function

$$f(x) = x \exp(-x/5).$$

Sketch the function using the first derivative test and confirm that $f \longrightarrow 0$ as $x \longrightarrow \infty$ so that the Nth term test does not predict divergence.

ii) Why can't we apply the integral test as it is stated in the notes?
iii) Consider

$$\sum_{n=1}^{\infty} n \exp(-n/5) = \sum_{n=1}^{5} n \exp(-n/5) + \sum_{n=6}^{\infty} n \exp(-n/5)$$

and explain why each of the two new pieces converges (use integral test on the second piece).

(Q3) It is also possible that the tests can be applied as written in the notes, but further work must be done to show convergence. Consider the series

$$\sum_{n=1}^{\infty} \exp(-n^2)$$

and use the integral test along with the comparison theorem for integrals to show that the series converges.

(Q4) Consider the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

i) Write out the first five terms.
ii) Show that the series converges for any $|x| < 1$. Use the comparison theorem.
iii) Show that the series converges for any x.
iv) Differentiate the series term by term and show that you get the negative of the series after the differentiation.
v) Use Maple to sketch the first five, first ten and first 100 terms for $0 \leq x \leq 5$.

(Q5) Consider the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

i) Write out the first five terms. If we wanted to write the series as a power series for all nonnegative integer powers of x what would the first five terms be?
ii) Show that the series converges for any $|x| < 1$. Use the comparison theorem.
iii) Show that the series converges for any x.
iv) Differentiate the series twice term by term and show that you get the negative of the original series after the differentiation.

(Q6) i) If you know S_N for all $N = 1, 2, 3, ..$ explain how you could determine a_n.
FOR ALL EXAMPLES FEEL FREE TO MODIFY THE LOWER BOUND OF THE SERIES FROM $n = 0$ to $n = 1$ OR ANY OTHER CONVENIENT FINITE NUMBER.
ii) Give an example of two series

$$\sum_{n=0}^{\infty} a_n$$

$$\sum_{n=0}^{\infty} b_n$$

that diverge, but for which

$$\sum_{n=0}^{\infty} a_n b_n$$

converges.
iii)Give an example of two series

$$\sum_{n=0}^{\infty} a_n$$

$$\sum_{n=0}^{\infty} b_n$$

that diverge, but for which

$$\sum_{n=0}^{\infty} a_n + b_n$$

converges.
iv) Give an example of two series

$$\sum_{n=0}^{\infty} a_n$$

which converges,

$$\sum_{n=0}^{\infty} b_n$$

that diverges, for which

$$\sum_{n=0}^{\infty} a_n b_n$$

diverges.
v) Repeat iv) but now find two series for which

$$\sum_{n=0}^{\infty} a_n b_n$$

converges.
vii)Give an example of two series

$$\sum_{n=0}^{\infty} a_n$$

converges and

$$\sum_{n=0}^{\infty} b_n$$

diverges, but for which

$$\sum_{n=0}^{\infty} a_n + b_n$$

diverges.
viii) For the case in vii) could

$$\sum_{n=0}^{\infty} a_n + b_n$$

ever converge?
(Q7) Series can be used to get solutions of differential equations.
Consider the initial value problem:

$$\frac{dy}{dx} = y$$

$y(0) = 5$.
Which has the solution $y(x) = 5\exp(x)$. Say we didn't know this and tried to find a series solution,

$$p(x) = \sum_{n=0}^{\infty} a_n x^n.$$

What does finding a solution mean in this case? It means finding all the various a_n.
i) Use the initial condition to find a_0.
ii) Substitute $p(x)$ into the DE (assume it's OK to differentiate term by term). You can do this using sums or just by writing out the first five or so terms in $p(x)$. You should find that

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ... = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

iii) Now since you already know a_0 from the initial conditions solve for a_1 in terms of a_0, a_2 in terms of a_1 and so on.
iv) Generalize your result in part iii) to get a_n in terms of a_{n-1}. This is called a *recursion relation*.
v) Get a general expression for a_n.

(Q8) Use the technique from Q7 to solve

$$\frac{dy}{dx} = -y$$

$y(0) = 1$. Your solution should confirm that the power series in Q4 is the one for a decaying exponential.

(Q9) i) Use the technique from Q7 to solve

$$x\frac{dy}{dx} + y = 0$$

$y(0) = 1$.
ii) Use the technique from Q7 to solve

$$\frac{dy}{dx} - 3\frac{y}{x} = 0$$

$y(0) = 0$. Is there anything special about the series?