MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 12

1: Let $z = e^{xy}\ln(y^2+1)$ where $x = s + 2t$ and $y = \dfrac{s}{t}$. Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $(s,t) = (-4,2)$.

Solution: Note that when $(s,t) = (-4,2)$ we have $(x,y) = (0,-2)$, and at this point we have

$$\frac{\partial z}{\partial x} = e^{xy}\, y\, \ln(y^2+1) = -2\ln 5\,,\quad \frac{\partial z}{\partial y} = x\, e^{xy}\, \ln(y^2+1) + e^{xy}\,\frac{2y}{y^2+1} = -\tfrac{4}{5}$$

$$\frac{\partial x}{\partial s} = 1\ ,\quad \frac{\partial x}{\partial t} = 2\,,\quad \frac{\partial y}{\partial s} = \frac{1}{t} = \tfrac{1}{2}\,,\quad \tfrac{\partial y}{\partial t} = -\tfrac{s}{t^2} = 1$$

and so

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (-2\ln 5)(1) + \left(-\tfrac{4}{5}\right)\left(\tfrac{1}{2}\right) = -2\ln 5 - \tfrac{2}{5}\ , \text{ and}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (-2\ln 5)(2) + \left(-\tfrac{4}{5}\right)(1) = -4\ln 5 - \tfrac{4}{5}\,.$$

2: Let $u = 4\,x\tan^{-1}\left(\dfrac{y}{z}\right)$ where $x = s^3 + t$, $y = \sqrt{s}\,t$ and $z = \dfrac{t}{s}$. Use the Chain Rule to find $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$ when $(s,t) = (1,-2)$.

Solution: Note that when $(s,t) = (1,-2)$ we have $(x,y,z) = (-1,-2,-2)$, and at this point we have

$$\frac{\partial u}{\partial x} = 4\tan^{-1}\frac{y}{z} = \pi\,,\quad \frac{\partial u}{\partial y} = \frac{4x}{1+(y/z)^2}\cdot\frac{1}{z} = 1\,,\quad \frac{\partial u}{\partial z} = \frac{4x}{1+\left(\frac{y}{z}\right)^2}\left(-\frac{y}{z^2}\right) = -1$$

$$\frac{\partial x}{\partial s} = 3s^2 = 3\,,\quad \frac{\partial x}{\partial t} = 1\,,\quad \frac{\partial y}{\partial s} = \frac{t}{2\sqrt{s}} = -1\,,\quad \frac{\partial y}{\partial t} = \sqrt{s} = 1\,,\quad \frac{\partial z}{\partial s} = -\frac{t}{s^2} = 2\,,\quad \frac{\partial z}{\partial t} = \frac{1}{s} = 1$$

and so

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = (\pi)(3) + (1)(-1) + (-1)(2) = 3\pi - 3\ , \text{ and}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = (\pi)(1) + (1)(1) + (-1)(1) = \pi$$

3: (a) Let $f(x,y)=y(xy+\ln x)$, $a=(1,-2)$ and $u=\left(\frac{4}{5},-\frac{3}{5}\right)$. Find $\nabla f(a)$ and $D_u f(a)$.

Solution: We have $f(x,y)=xy^2+y\ln x$, so $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=\left(y^2+\frac{y}{x},2xy+\ln x\right)$ and so we have $\nabla f(a)=\nabla f(1,-2)=(2,-4)$ and $D_u f(a)=(2,-4)\bullet\left(\frac{4}{5},-\frac{3}{5}\right)=\frac{8}{5}+\frac{12}{5}=4$

(b) Let $g(x,y,z)=\dfrac{x}{y+z^2}$, $b=(2,1,-1)$ and $v=\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$. Find $\nabla g(b)$ and $D_v g(b)$.

Solution: We have $\nabla g=\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z}\right)=\left(\frac{1}{y+z^2},\frac{-x}{(y+z^2)^2},\frac{-2xz}{(y+z^2)^2}\right)$, so $\nabla g(b)=\nabla g(2,1,-1)=\left(\frac{1}{2},-\frac{1}{2},1\right)$ and $D_v g(b)=\left(\frac{1}{2},-\frac{1}{2},1\right)\bullet\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)=\frac{1}{3}+\frac{1}{6}+\frac{2}{3}=\frac{7}{6}$

4: A boy is standing on a hill whose shape is given by $z=\dfrac{600}{100+4x^2+y^2}$ at the point $(5,10,2)$ (where x, y and z are in meters).

(a) At the point where the boy is standing, in which direction is the slope steepest?

Solution: Write $z=f(x,y)=\frac{600}{100+4x^2+y^2}$ and $a=(5,10)$. Then $\nabla f=\left(\frac{-4800\,x}{(100+4x^2+y^2)^2},\frac{-1200\,y}{(100+4x^2+y^2)^2}\right)$ and so $\nabla f(a)=\left(-\frac{24{,}000}{900{,}000},-\frac{12{,}000}{900000}\right)=\left(-\frac{4}{15},-\frac{2}{15}\right)=\frac{2}{15}(-2,-1)$. Thus the slope is the steepest in the direction of the unit vector $\frac{1}{\sqrt{5}}(-2,-1)$.

(b) If the boy walks southeast, then will he be ascending or descending?

Solution: The southeasterly direction is in the direction of the unit vector $v=\frac{1}{\sqrt{2}}(1,-1)$, and we have $D_v f(a)=\frac{2}{15\sqrt{2}}(-2,-1)\bullet(1,-1)=-\frac{\sqrt{2}}{15}<0$, so the boy would be descending.

(c) If the boy walks in the direction of steepest slope, then at what angle (from the horizontal) will he be climbing?

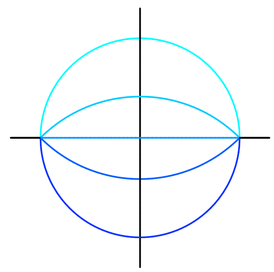
Solution: If the boy walks in the direction of the unit vector $u=\frac{1}{|\nabla f(a)|}\nabla f(a)$, then the slope in that direction is $D_u f(a)=|\nabla f|=\frac{2}{15}\big|(-2,-1)\big|=\frac{2\sqrt{5}}{15}$, so the angle of ascent is $\theta=\tan^{-1}\frac{2\sqrt{5}}{15}\cong 16.6°$.

5: The temperature around the outer circle of a metal disc of radius 1 meter is held constant, with the top half of the circle held at 0° C and the bottom half of the circle held at 20° C. It can be shown that the temperature at all points of the disc is given by

$$T(x,y) = 10 + \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right).$$

(a) Sketch the isotherms (level curves of constant temperature) $T = 0, 5, 10, 15, 20$.

Solution: We know that $T = 0$ along the top half of the boundary circle, and $T = 20$ along the bottom half. We have $T = 5 \iff \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = -5 \iff \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = -\frac{\pi}{4} \iff \frac{2y}{x^2+y^2-1} = -1 \iff 2y = -x^2-y^2+1 \iff x^2+y^2+2y = 1 \iff x^2+(y+1)^2 = 2$, and this is the circle of radius $\sqrt{2}$ centered at $(0,-1)$. Similarly, we have $T = 15 \iff x^2+y^2-2y = 1 \iff x^2+(y-1)^2 = 2$, and this is the circle of radius $\sqrt{2}$ centered at $(0,1)$. Also, we have $T = 10 \iff \frac{20}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = 0 \iff \frac{2t}{x^2+y^2-1} = 0 \iff 2y = 0 \iff y = 0$. These isotherms are shown below.



(b) Find $T\left(\frac{1}{2},\frac{1}{2}\right)$ and $\nabla T\left(\frac{1}{2},\frac{1}{2}\right)$.

Solution: We have $T\left(\frac{1}{2},\frac{1}{2}\right) = 10 + \frac{20}{\pi}\tan^{-1}\left(\frac{1}{-1/2}\right) = 10 - \frac{20}{\pi}\tan^{-1}2 \cong 2.95$. Also, we have

$$\begin{aligned}\nabla T(x,y) = \left(\frac{\partial T}{\partial x},\frac{\partial T}{\partial y}\right) &= \frac{20}{\pi}\left(\frac{1}{1+\left(\frac{2y}{x^2+y^2-1}\right)^2}\cdot\frac{-4xy}{(x^2+y^2-1)^2},\ \frac{1}{1+\left(\frac{2y}{x^2+y^2-1}\right)^2}\cdot\frac{2(x^2+y^2-1)-4y^2}{(x^2+y^2-1)^2}\right)\\ &= \frac{20}{\pi}\left(\frac{-4xy}{(x^2+y^2-1)^2+4y^2},\ \frac{2(x^2-y^2-1)}{(x^2+y^2-1)^2+4y^2}\right)\end{aligned}$$

and so $\nabla T\left(\frac{1}{2},\frac{1}{2}\right) = \frac{20}{\pi}\left(\frac{-1}{\frac{1}{4}+1},\frac{-2}{\frac{1}{4}+1}\right) = \frac{20}{\pi}\left(-\frac{4}{5},-\frac{8}{5}\right) = \frac{16}{\pi}(-1,-2)$.

(c) Find the equation of the tangent line to the isotherm through $\left(\frac{1}{2},\frac{1}{2}\right)$.

Solution: The tangent line to this isotherm is perpendicular to $\nabla T\left(\frac{1}{2},\frac{1}{2}\right) = \frac{16}{\pi}(-1,-2)$, so it is in the direction of the vector $(2,-1)$. A vector equation for the tangent line is $(x,y) = \left(\frac{1}{2},\frac{1}{2}\right) + t\,(2,-1)$.

(d) Show that if an ant starts at the point $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (where the temperature is 0°) and walks on the disc in the direction of ∇T (that is, in the direction in which the temperature increases most rapidly), then it will follow an arc of the circle of radius $\sqrt{3}$ centered at $(2,0)$.

Solution: By the formula for $\nabla T(x,y)$ that we found in part (b), we see that at each point $\nabla T(x,y)$ is in the direction of the vector $\big(-4xy, 2(x^2-y^2-1)\big)$, which has slope $\frac{2(x^2-y^2-1)}{-4xy} = \frac{-x^2+y^2+1}{2xy}$. The circle of radius $\sqrt{3}$ centered at $(2,0)$ has equation $(x-2)^2+y^2 = 3$. Differentiating implicitly gives $2(x-2)+2y\,y' = 0$ and so $y' = \frac{2-x}{y}$. For any point (x,y) on this circle, we have $(x-2)^2+y^2 = 3 \Longrightarrow x^2-4x+4+y^2 = 3 \Longrightarrow 4x = x^2+y^2+1$, and so at such a point $y' = \frac{2-x}{y} = \frac{4x-2x^2}{2xy} = \frac{(x^2+y^2+1)-2x^2}{2xy} = \frac{-x^2+y^2+1}{2xy}$, which is the same as the slope of the gradient vector.