MATH 128 = Calculus 2 for the Sciences, Fall 2006

Assignment 2

Due September 27 before class

Determine whether each integral is convergent or divergent. Evaluate those that are convergent. State clearly in the final line of your solution whether the integral is convergent or divergent.

Justify each answer, showing all necessary steps. Simplify your answers as much as possible (without a calculator, of course).

1. $\int_{-\infty}^{\frac{\pi}{2}} \sin 2t dt$
2. $\int_0^5 \frac{4}{(r-4)^3} dr$
3. $\int_{\cancel{0}\,1}^{\infty} \frac{\ell n\, x}{x^4} dx$
4. $\int_0^3 \frac{a}{a^2+a-12} da$
5. $\int_{-\infty}^{\infty} \frac{4x^3}{(1+x^4)^2} dx$
6. $\int_{-3}^{3} \frac{\pi}{\sqrt{9-x^2}} dx$
7. An oil field is estimated to produce oil at a rate given by $f(t) = 600e^{-0.1t}$ thousand barrels per month, t months into production. To estimate the total amount of oil produced by this oil well, we integrate the proposed rate function, $f(t)$. Thus, the total amount of oil produced by the oil well in T months of production is

$$\phi(T) = \int_0^T 600e^{-0.1t} dt.$$

The potential output of the well can be determined by letting $T \to \infty$, whereby we assume the well runs indefinitely.

Calculate the potential output of oil, $\lim_{T\to\infty} \phi(T)$. Include units in your final answer.

Total (/65)

MATH 128 FALL 2006 ASSIGNMENT 2 - SOLUTIONS

(/5) 1. $I = \int_{-\infty}^{\frac{\pi}{2}} \sin 2t\,dt = \lim_{a\to-\infty} \int_a^{\frac{\pi}{2}} \sin 2t\,dt$ (1)

$= \lim_{a\to-\infty} \left[-\frac{\cos 2t}{2}\right]_a^{\pi/2}$ (1)

$= \lim_{a\to-\infty} \left[-\frac{\cos\pi}{2} + \frac{\cos 2a}{2}\right]$ (1)

(1) Since $\lim_{a\to-\infty} \cos 2a$ does not exist, the integral

$\therefore$ is $\boxed{\text{divergent.}}$ (1)

(/9) 2. $I = \int_0^5 \frac{4\,dr}{(r-4)^3} = 4\int_0^4 \frac{dr}{(r-4)^3} + 4\int_4^5 \frac{dr}{(r-4)^3}$

$= 4\lim_{b\to4^-} \int_0^b \frac{dr}{(r-4)^3} + 4\lim_{a\to4^+} \int_a^5 \frac{dr}{(r-4)^3}$ (2)

$= 4\lim_{b\to4^-} \left[\frac{-1/2}{(r-4)^2}\right]_0^b + 4\lim_{a\to4^+} \left[\frac{-1/2}{(r-4)^2}\right]_a^5$ (2)

$= -2\lim_{b\to4^-} \left[\frac{1}{(b-4)^2} - \frac{1}{16}\right] - 2\lim_{a\to4^+} \left[1 - \frac{1}{(a-4)^2}\right]$ (2)

either term approaches ∞ (1)

(1) $\Rightarrow$ limit does not exist,

$\therefore$ the integral is $\boxed{\text{divergent}}$ (1)

3. $I = \int_1^\infty \frac{\ln x}{x^4}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^4}\,dx$ 1

/14

IBP

$$\left.\begin{array}{ll} u = \ln x & v = -\frac{1}{3x^3} \\ du = \frac{1}{x}\,dx & dv = \frac{1}{x^4}\,dx \end{array}\right\}$$ 2

$$\Rightarrow I = \lim_{b\to\infty}\left[\left[-\frac{\ln x}{3x^3}\right]_1^b + \frac{1}{3}\int_1^b \frac{dx}{x^4}\right]$$ 2

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{3b^3} + \underbrace{\frac{\cancel{\ln 1}}{3}}_{0} + \frac{1}{3}\left[-\frac{1}{3x^3}\right]_1^b\right]$$ 2

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{3b^3} - \frac{1}{9}\left(\underbrace{\frac{1}{b^3}}_{\to 0} - 1\right)\right]$$ 2

2 $$= \frac{1}{9} - \frac{1}{3}\lim_{b\to\infty}\frac{1/b}{3b^2}$$ (l'Hospital's Rule)

$$= \frac{1}{9} - \frac{1}{9}\underbrace{\lim_{b\to\infty}\frac{1}{b^3}}_{\to 0}$$ 1

$\therefore$ $I = \boxed{\frac{1}{9}}$, the integral is $\boxed{\text{convergent}}$.

1 1

/13

4. $I = \int_0^3 \frac{a}{a^2+a-12}\,da = \int_0^3 \frac{a}{(a-3)(a+4)}\,da$ 1

$= \lim_{b\to 3^-} \int_0^b \frac{a}{(a-3)(a+4)}\,da$ 1

$= \lim_{b\to 3^-} \int_0^b \left(\frac{A}{a-3} + \frac{B}{a+4}\right) da$ (partial fractions)

$= \lim_{b\to 3^-} \int_0^b \frac{A(a+4)+B(a-3)}{(a-3)(a+4)}\,da$

$\Rightarrow a = A(a+4) + B(a-3)$

$\Rightarrow a = Aa + 4A + Ba - 3B = a(A+B) + 4A - 3B$

$\Rightarrow \begin{cases} 1 = A + B \\ 0 = 4A - 3B \end{cases} \Rightarrow \quad 4 = 4A + 4B$

$\quad\quad -(0 = 4A - 3B)$

$\quad\quad 4 = 7B \Rightarrow \boxed{B = \frac{4}{7}}$

$1 = A + B \Rightarrow 1 = A + \frac{4}{7} \Rightarrow \boxed{A = \frac{3}{7}}$

4
1 - A
1 - B
2 - work

$\Rightarrow I = \frac{1}{7} \lim_{b\to 3^-} \int_0^b \left(\frac{3}{a-3} + \frac{4}{a+4}\right) da$ 1

$= \frac{1}{7} \lim_{b\to 3^-} \Big[3\ln|a-3| + 4\ln(a+4) \Big]_0^b$ 2

2 $= \frac{1}{7} \lim_{b\to 3^-} \Big[\underbrace{3\ln|b-3|}_{-\infty} + 4\ln(b+4) - 3\ln|0-3| - 4\ln 4 \Big]$ 1

$\therefore$ I is $\boxed{\text{divergent}}$ 1 or some other constant

/9

5. $I = \int_{-\infty}^{\infty} \frac{4x^3}{(1+x^4)^2}\,dx = \int_{-\infty}^{0} \frac{4x^3\,dx}{(1+x^4)^2} + \int_{0}^{\infty} \frac{4x^3}{(1+x^4)^2}\,dx$

$= \lim_{a \to -\infty} \int_a^0 \frac{4x^3\,dx}{(1+x^4)^2} + \lim_{b \to \infty} \int_0^b \frac{4x^3\,dx}{(1+x^4)^2}$ 2

optional (may do integration directly)

Let $u = 1+x^4$ (or $u = x^4$ will work)

$du = 4x^3dx$

$\Rightarrow \int \frac{4x^3\,dx}{(1+x^4)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{1+x^4}$

$\Rightarrow I = \lim_{a \to -\infty} \left[-\frac{1}{1+x^4}\right]_a^0 + \lim_{b \to \infty} \left[-\frac{1}{1+x^4}\right]_0^b$ 2

$= \lim_{a \to -\infty} \left[-1 + \underbrace{\frac{1}{1+a^4}}_{0}\right] + \lim_{b \to \infty} \left[\underbrace{-\frac{1}{1+b^4}}_{0} + 1\right]$ 2

1 $= -1 + 1$

$\therefore$ $\boxed{I = 0}$, and the integral is $\boxed{\text{convergent}}$.

1 1

or some other constant

6\. $I = \int_{-3}^{3} \frac{\pi\, dx}{\sqrt{9-x^2}} = \int_{-3}^{0} \frac{\pi\, dx}{\sqrt{9-x^2}} + \int_{0}^{3} \frac{\pi\, dx}{\sqrt{9-x^2}}$ (2)

10

$= \lim_{a\to -3^+} \int_{a}^{0} \frac{\pi\, dx}{\sqrt{9-x^2}} + \lim_{b\to 3^-} \int_{0}^{b} \frac{\pi\, dx}{\sqrt{9-x^2}}$

(3)

let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\Rightarrow \int \frac{\pi\, dx}{\sqrt{9-x^2}} = \pi \int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \pi \int d\theta = \pi\,\theta$

$= \pi \sin^{-1}\frac{x}{3}$

$\Rightarrow I = \lim_{a\to -3^+} \left[\pi \sin^{-1}\frac{x}{3}\right]_a^0 + \lim_{b\to 3^-} \left[\pi \sin^{-1}\frac{x}{3}\right]_0^b$

(2) $= \lim_{a\to -3^+} \left[\underset{0}{\pi \sin^{-1} 0} - \underset{-\frac{\pi}{2}}{\pi \sin^{-1}\frac{a}{3}}\right] + \lim_{b\to 3^-} \left[\underset{\frac{\pi}{2}}{\pi \sin^{-1}\frac{b}{3}} - \underset{0}{\pi \sin^{-1} 0}\right]$

(1) $= \frac{\pi^2}{2} + \frac{\pi^2}{2}$

$\therefore \boxed{I = \pi^2}$, the integral is $\boxed{\text{convergent}}$!

(1) (1)

5

7. $$\lim_{T\to\infty}\phi(T) = \lim_{T\to\infty}\int_0^T 600e^{-0.1t}\,dt = \lim_{T\to\infty}\left[\frac{600}{-0.1}e^{-0.1t}\right]_0^T$$

$$= \lim_{T\to\infty}\left[-6000e^{-0.1t}\right]_0^T \quad 1$$

$$= \lim_{T\to\infty}\left(-6000e^{-0.1T} + 6000e^{-0.1(0)}\right) \quad 1$$

$$= \lim_{T\to\infty}\left(\underbrace{-6000e^{-0.1T}}_{0} + 6000\right) \quad 1$$

$$= \boxed{6000} \quad 1$$

$\therefore$ The total production of the oil well is estimated to be 6000 thousand, or $\boxed{6{,}000{,}000 \text{ barrels.}}$ 1