MATH 128 Calculus 2, Solutions to Term Test 2, Winter 2007

[10] **1:** (a) Let $a_1 = 1$ and for $n \geq 1$ let $a_{n+1} = 2\sqrt{a_n}$. Show that the sequence $\{a_n\}$ converges, and find the limit.

Solution: If the sequence does converge, with say $\lim_{n\to\infty} a_n = l$, then by taking the limit on both sides of the equation $a_{n+1} = 2\sqrt{a}_n$ we find that $l = 2\sqrt{l}$, so $l^2 = 4l$ and so $l = 0$ or $l = 4$. The first few terms of the sequence are $a_1 = 1$, $a_2 = 2$ and $a_3 = 2\sqrt{3}$. We claim that $0 < a_n < a_{n+1} < 4$ for all $n \geq 1$. Note that we do have $0 < a_1 < a_2 < 4$. Suppose that $0 < a_k < a_{k+1} < 4$. Then $0 < \sqrt{a_k} < \sqrt{a_{k+1}} < 2$ and so $0 < 2\sqrt{a_k} < 2\sqrt{a_{k+1}} < 4$, that is $0 < a_{k+1} < a_{k+2} < 4$. Thus the claim is true for all $n \geq 1$, by induction. This shows that $\{a_n\}$ is increasing and bounded above by 4, so it does converge. Since it converges, the limit must be 0 or 4 (as shown above). Since $a_1 = 1$ and $\{a_n\}$ increases, we must have $\lim_{n\to\infty} a_n = 4$.

(b) Evaluate the sum $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1} - 2}{5^n}$.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1} - 2}{5^n} = \sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n} - \sum_{n=1}^{\infty} \frac{2}{5^n} = \frac{\frac{9}{5}}{1 - \frac{3}{5}} - \frac{\frac{2}{5}}{1 - \frac{1}{5}} = \frac{9}{2} - \frac{1}{2} = 4.$

[10] **2:** Test each of the following series for convergence. Indicate which convergence tests you use for each series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$

Solution: Let $a_n = \dfrac{n}{\sqrt{n^3+2}}$ and let $b_n = \dfrac{n}{\sqrt{n^3}} = \dfrac{1}{n^{1/2}}$. Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum b_n$ diverges, so $\sum a_n$ diverges too, by the L.C.T.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{2}$

Solution: Let $a_n = (-1)^n \sqrt[n]{2}$. Then $|a_n| = \sqrt[n]{2} = 2^{1/n}$. As $n \to \infty$ we have $\frac{1}{n} \to 0$ so $|a_n| \to 2^0 = 1$. Since $|a_n| \not\to 0$, we have $a_n \not\to 0$ so $\sum a_n$ diverges by the D.T.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution: Let $a_n = \dfrac{1}{n \ln n}$ and let $f(x) = \dfrac{1}{x \ln x}$ so that $a_n = f(n)$. Note that $f(x)$ is decreasing for $n \geq 2$ so we can apply the I.T. Letting $u = \ln x$ so that $du = \frac{1}{x}\,dx$ we have $\displaystyle\int f(x)\,dx = \int \frac{dx}{x\,\ln x} = \int \frac{du}{u} = \ln u + c = \ln(\ln x) + c$, so $\displaystyle\int_2^{\infty} f(x) = \Big[\ln(\ln x)\Big]_2^{\infty} = \infty$. Thus $\sum a_n$ diverges by the I.T.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2^{(n^2)}}$

Solution: Let $a_n = \dfrac{n!}{2^{n^2}}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{2^{n^2+2n+1}} \cdot \dfrac{2^{n^2}}{n!} = \dfrac{n+1}{2^{2n+1}} = \dfrac{n+1}{2 \cdot 4^n}$. By l'Hôpital's Rule, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{x\to\infty} \frac{x+1}{2 \cdot 4^x} = \lim_{x\to\infty} \frac{1}{2 \cdot \ln 4 \cdot 4^x} = 0$. Thus $\sum a_n$ converges by the R.T.

[10] **3:** (a) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(3-2x)^n}{\sqrt{n}}$.

Solution: Let $a_n=\dfrac{(3-2x)^n}{\sqrt{n}}$. Then $\dfrac{|a_{n+1}|}{|a_n|}=\dfrac{|3-2x|^{n+1}}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{|3-2x|}=\sqrt{\frac{n}{n+1}}\,|3-2x|$, so $\displaystyle\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|3-2x|=|2x-3|$. By the R.T, $\sum a_n$ converges when $|2x-3|<1$ and diverges when $|2x-3|>1$. Note that $|2x-3|<1 \iff -1<2x-3<1 \iff 2<2x<4 \iff 1<x<2$. When $x=1$ we have $a_n=\frac{1}{\sqrt{n}}$ so $\sum a_n$ diverges, and when $n=2$ we have $a_n=\frac{(-1)^n}{\sqrt{n}}$, so $\sum a_n$ converges by the A.S.T. Thus the interval of convergence is $(1,2]$.

(b) Find the Taylor polynomial of degree 3 centered at $x=0$ for $f(x)=\dfrac{\ln(1+x)}{e^x}$.

Solution: We have

$$\begin{aligned} f(x)&=\ln(1+x)e^{-x}\\ &=\left(x-\tfrac{1}{2}\,x^2+\tfrac{1}{3}\,x^3-\cdots\right)\left(1-x+\tfrac{1}{2}\,x^2-\cdots\right)\\ &=x+\left(-1-\tfrac{1}{2}\right)x^2+\left(\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{3}\right)x^3+\cdots\\ &=x-\tfrac{3}{2}\,x^2+\tfrac{4}{3}\,x^3+\cdots\,. \end{aligned}$$

Thus the Taylor polynomial is $T_3(x)=x-\frac{3}{2}\,x^2+\frac{4}{3}\,x^3$.

[10] **4:** (a) Let $f(x)=\cos(x^2/2)$. Find the 8^{th} derivative $f^{(8)}(0)$.

Solution: We have $\cos x = 1-\frac{1}{2!}\,x^2+\frac{1}{4!}\,x^4-\cdots$ so

$$\begin{aligned}\cos\left(\tfrac{x^2}{2}\right) &= 1-\tfrac{1}{2!}\left(\tfrac{x^2}{2}\right)^2+\tfrac{1}{4!}\left(\tfrac{x^2}{2}\right)^4-\cdots\\ &= 1-\tfrac{1}{2^2\,2!}\,x^4+\tfrac{1}{2^4\,4!}\,x^8-\cdots\end{aligned}$$

Thus $f^{(8)}(0)=8!\,c_8=\frac{8!}{2^4\,4!}=\frac{8\cdot7\cdot6\cdot5}{2^4}=7\cdot3\cdot5=105$.

(b) Evaluate the sum $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{2^n n!}$. Hint: use the Taylor series centered at 0 for $f(x)=x\,e^x$.

Solution: We have $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\,x^n$ so $x\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\,x^{n+1}$. Take the derivative on both sides to get $(x+1)\,e^x=\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{n!}\,x^n$. Put in $x=\frac12$ to get $\frac32\,e^{1/2}=\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{2^n\,n!}$.

[10] **5:** Estimate the value of $\frac{1}{\sqrt{30}}$ so that the absolute error is $E \leq \frac{1}{10{,}000}$.
Hint: use the Taylor series centered at 0 for $f(x) = (25+x)^{-1/2}$.

Solution: We have

$$\begin{aligned} f(x) &= (25+x)^{-1/2} \\ &= \tfrac{1}{5}\left(1+\tfrac{x}{25}\right)^{-1/2} \\ &= \tfrac{1}{5}\left(1 + \left(-\tfrac{1}{2}\right)\left(\tfrac{x}{25}\right) + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\tfrac{x}{25}\right)^2 + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\tfrac{x}{25}\right)^3 + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(\tfrac{x}{25}\right)^4 + \cdots\right) \end{aligned}$$

for all x with $\left|\frac{x}{25}\right| < 1$, that is all x with $|x| < 25$, and so

$$\begin{aligned} \tfrac{1}{\sqrt{30}} &= f(5) \\ &= \tfrac{1}{5}\left(1 + \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{5}\right) + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\tfrac{1}{5}\right)^2 + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\tfrac{1}{5}\right)^3 + \tfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(\tfrac{1}{5}\right)^4 + \cdots\right) \\ &= \tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} + \tfrac{1\cdot 3}{2^2\cdot 2!\cdot 5^3} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 3!\cdot 5^4} + \tfrac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!\cdot 5^5} - \cdots \\ &\cong \tfrac{1}{5} - \tfrac{1}{2\cdot 5^2} + \tfrac{1\cdot 3}{2^2\cdot 2!\cdot 5^3} - \tfrac{1\cdot 3\cdot 5}{2^3\cdot 3!\cdot 5^4} = \tfrac{1}{5} - \tfrac{1}{50} + \tfrac{3}{1000} - \tfrac{1}{2000} = \tfrac{400-40+6-1}{2000} = \tfrac{365}{2000} = \tfrac{73}{400} \end{aligned}$$

with absolute error $E \leq \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!\cdot 5^5} = \frac{7}{80{,}000} < \frac{1}{10{,}000}$ by the A.S.T.

To be completely rigorous, we should verify that the A.S.T. can be applied. To do this, let $a_n = \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!\cdot 5^{n+1}}$ so that $\frac{1}{\sqrt{30}} = 1 + \sum_{n=1}^{\infty} a_n$. We already know that the sum converges, and so we know that $a_n \to 0$ (and hence $|a_n| \to 0$) by the D.T, but we also need to check that $\{|a_n|\}$ is decreasing in order to be able to apply the A.S.T. Note that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}(n+1)!\,5^{n+2}} \cdot \frac{2^n n!\,5^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2n+1}{2\cdot 5\cdot(n+1)} < \frac{2(n+1)}{2\cdot 5\cdot(n+1)} = \frac{1}{5}$$

and so we do have $|a_{n+1}| < |a_n|$ for all n, as required.