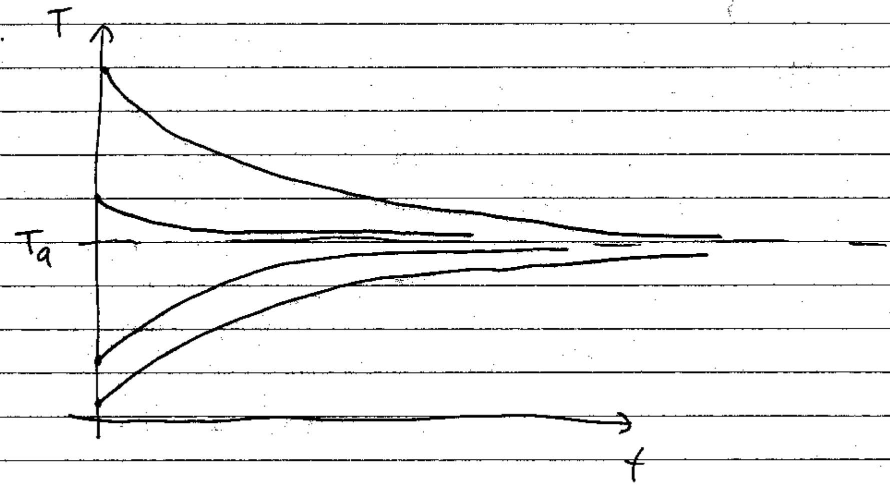
Assignment 4

Q1 iii)

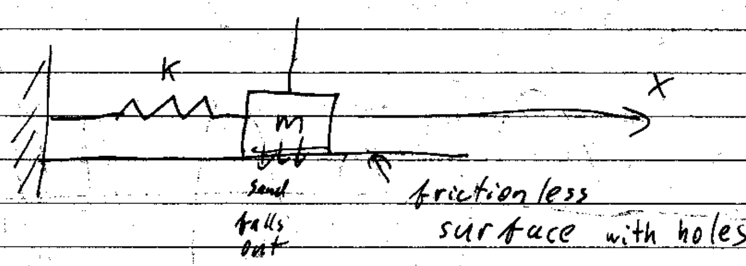


$$T(t) = \underbrace{T_a}_{\text{equilibrium}} + \underbrace{(T_0 - T_a)}_{\text{departure from equilibrium}} e^{-kt}$$

k — rate of "equilibriation"

Q1 i) & ii) are on page 35 in Course materials

Q2 i)



Gravity is balanced by reaction force so the only force is $F_{spring} = -k\,x(t)$

Newton's 2nd law $m(t)\,a(t) = F_{spring}$

or $m(t)\dfrac{d^2x}{dt^2} = -k\,x(t)$

$$\boxed{\frac{d^2x(t)}{dt^2} + \frac{k}{m(t)}x(t) = 0}$$

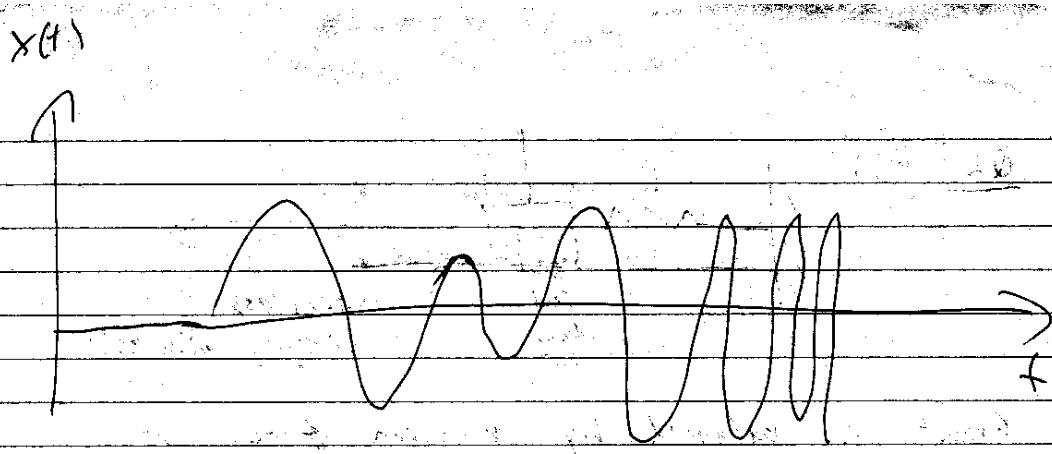
ii) If $m(t)$ changes slowly then at any one time it is as if we were solving

$$\frac{d^2x(t)}{dt^2} + \frac{k}{m}x(t) = 0$$

which has solutions $A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)$

as m decreases the frequency $\omega = \sqrt{\frac{k}{m}}$ increases

so the oscillations get faster and faster


x(t)
t

Q3) i) Start with

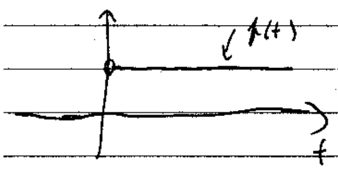
$$\frac{dx(t)}{dt} = \frac{1}{k}\frac{df(t)}{dt} + \frac{1}{D} f(t)$$

take Laplace transforms to get

$$s\bar{X}(s) = \frac{1}{k} s F(s) + \frac{1}{D} F(s)$$

where I used $x(0) = f(0) = 0$

Now $f(t) = H(t)$ means $F(s) = \frac{1}{s}$ and



so

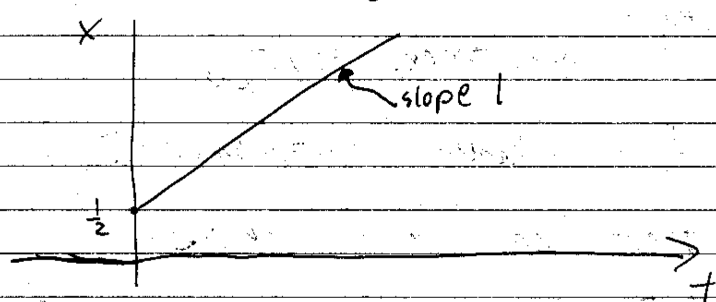
$$\bar{X}(s) = \frac{1}{k}\frac{1}{s} + \frac{1}{D}\frac{1}{s^2}$$

and from our table of Laplace

transforms $x(t) = \frac{1}{k} H(t) + \frac{t}{D}$

when $D = 1$ $k = 2$ get

$$x(t) = \frac{1}{2} H(t) + t$$



ii) Start with the Laplace transformed equations:

$$s\bar{X}(s) = \left(\frac{s}{k} + \frac{1}{D}\right) F(s)$$

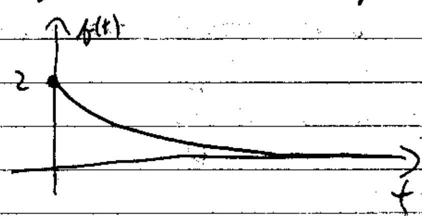
If $x(t) = H(x)$ then $\bar{X}(s) = \frac{1}{s}$ and

$$F(s) = \frac{1}{\frac{s}{k} + \frac{1}{D}} = k \frac{1}{s + \frac{k}{D}}$$

Using our table we find

$$f(t) = k \exp\left\{-\frac{k}{D} t\right\}$$

If $D = 1$ $k = 2$ $f(t) = 2e^{-2t}$



iii) It is easier to start with part ii)
Here we write

$$f(t) = k \exp\{-t/T_R\}$$

where $T_R = \frac{D}{k}$ [harder spring => smaller T_R]

For part i) there are many choices

$T_R = 0, \infty$ and D from the jump, the no upper bound & $\frac{1}{\text{slope}}$

(Q4) i) $x'(t) - x(t) = 1 - t$

$$p'(t) = -1 \Rightarrow p(t) = -t$$

Multiply by $e^{p(t)} = e^{-t}$ to get

$$e^{-t}\frac{dx}{dt} - e^{-t}x(t) = e^{-t}(1-t)$$

but the LHS is just

$$\frac{d}{dt}\left(x(t)e^{-t}\right) \quad \text{so that } x(t)e^{-t} = \int \text{RHS } dt$$

and we need to find the anti-derivative of $e^{-t}(1-t)$

$$\int (e^{-t} - te^{-t})\,dt = \int \left(\frac{dt}{dt}e^{-t} + t\frac{de^{-t}}{dt}\right)dt$$

$$= \int \frac{d}{dt}(te^{-t})\,dt = te^{-t} + C$$

If you don't see it, you could also just apply integration by parts to te^{-t}.
After integrating we have

$$x(t)e^{-t} = te^{-t} + C \Rightarrow x(t) = t + Ce^{t}$$

If $x(0)=0$ must have $C=0$ so $\boxed{x(t) = t}$

Q4 ii) This is just like part i) up to

$$x(t) = t + Ce^t$$

Now, however $x(0)=3$ so that $Ce^0 = 3$

$\therefore C=3$ and $\boxed{x(t) = t + 3e^t}$

Q4 iii)

$$x'(t) - 2t\, x(t) = 1 - 2t^2$$

$$p'(t) = -2t \Rightarrow p(t) = -t^2 \quad \text{so multiply by } e^{-t^2}$$

to get

$$e^{-t^2}\frac{dx}{dt} + x(t)\frac{de^{-t^2}}{dt} = e^{-t^2}(1-2t^2)$$

$$\frac{d}{dt}\left(e^{-t^2}x(t)\right) = e^{-t^2}(1-2t^2)$$

To integrate the RHS notice

$$\frac{d}{dt}\left[te^{-t^2}\right] = e^{-t^2} + t\frac{d}{dt}(e^{-t^2}) = e^{-t^2} + t\,e^{-t^2}(-2t)$$

$$= e^{-t^2}[1-2t^2]$$

Thus

$$e^{-t^2}x(t) = te^{-t^2} + C$$

$$\boxed{x(t) = t + Ce^{t^2}}$$

$x(0)=0 \Rightarrow C=0 \quad \therefore \quad \boxed{x(t)=t}$

Q4 iv) This works just like part iii so

$$x(t) = t + Ce^{t^2} \quad \text{and} \quad x(0) = 2$$

means

$$C = 2 \Rightarrow \boxed{x(t) = t + 2e^{t^2}}$$

Q4 v) $$x'(t) + \cos(t)\, x(t) = \frac{\cos(t)}{t} - \frac{1}{t^2}$$

$p'(t) = \cos(t) \Rightarrow p(t) = \sin t$ so we multiply by $e^{\sin t}$ to get

$$e^{\sin(t)}\frac{dx}{dt} + e^{\sin(t)}\cos(t)\, x(t) = e^{\sin(t)}\left[\frac{\cos t}{t} - \frac{1}{t^2}\right]$$

$$\frac{d}{dt}\left(e^{\sin(t)} x(t)\right) = e^{\sin(t)}\left\{\frac{\cos t}{t} - \frac{1}{t^2}\right\}$$

Using Maple I get

$$e^{\sin(t)} x(t) = \exp\left\{\frac{2\tan\left(\frac{1}{2}t\right)}{1+\tan^2\left(\frac{1}{2}t\right)}\right\} + C$$

$$\boxed{x(t) = \frac{e^{-\sin(t)}}{t}\exp\left\{\frac{2\tan(t/2)}{1+\tan^2(t/2)}\right\} + Ce^{-\sin(t)}}$$

If $x(2) = 0$ then

$$C = -\frac{1}{2}e^{\frac{2\tan(1)}{1+\tan(1)^2}} = -\frac{1}{2}e^{8\pi/(16+\pi^2)}$$

because $\tan(1) = \frac{\pi}{4}$

Q4 vi) Once again

$$x(t) = \frac{e^{-\sin(t)}}{t} \exp\left\{ \frac{2\tan(t/2)}{1+\tan^2(t/2)} \right\} + Ce^{-\sin(t)}$$

$x(2) = -1$ gives

$$\boxed{C = -\frac{3}{2} e^{\sin(2)}}$$

The point of these last two parts is that even with very nasty integrals some tidy answers can result, though you should wonder if the pain is worth it; what does the formula say? Hard to tell without graphing, which we could get from a numerical calculation.

The reason we can't give a condition at $t=0$ is because the rate of change, $\frac{dx}{dt}$, is <u>undefined</u> when $t=0$.

Q5 i) $\frac{d^2x(t)}{dt^2} + x(t) = 0$

If we write $\frac{dx(t)}{dt} = v(t)$ then

$\frac{d^2x(t)}{dt^2} = \frac{dv}{dt}$ and so the SHO can

be written as
$$\begin{cases} \frac{dv}{dt} = -x(t) \\ \frac{dx(t)}{dt} = v(t) \end{cases}$$

ii) The Laplace transform gives

$$s\,V(s) - v(0) = -X(s)$$

$$s\,X(s) - x(0) = V(s)$$

Now $x(0) = 2$ and $v(0) = -1$ so that

$$s\,V(s) + X(s) = -1$$

$$-V(s) + s\,X(s) = 2$$

Next I write it as a matrix

$$\begin{pmatrix} s & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} V(s) \\ X(s) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ and row reduce

$$\left(\begin{array}{cc|c} s & 1 & -1 \\ -1 & s & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ -1 & s & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ 0 & \frac{1}{s}+s & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ 0 & 1 & 2\frac{s}{s^2+1} \end{array}\right)$$

so $X(s) = 2\dfrac{s}{s^2+1}$

$$V(s) + \frac{1}{s}X(s) = -\frac{1}{s}$$

$$V(s) = -\frac{1}{s} - 2\frac{1}{s^2+1}$$

Q5 ii) cont'd

$$\left(\begin{array}{cc|c} s & 1 & -1 \\ -1 & s & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ -1 & s & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ 0 & s+\frac{1}{s} & 2-\frac{1}{s} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ 0 & \frac{s^2+1}{s} & \frac{2s-1}{s} \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1/s & -1/s \\ 0 & 1 & \frac{2s-1}{s^2+1} \end{array}\right) \to \left(\begin{array}{cc|c} s & 1 & -1 \\ 0 & 1 & \frac{2s-1}{s^2+1} \end{array}\right)$$

$$X(s) = \frac{2s-1}{s^2+1} = 2\frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\boxed{x(t) = 2\cos(t) - \sin(t)}$$

$$\to \left(\begin{array}{cc|c} s & 0 & -1 - \frac{2s-1}{s^2+1} \\ 0 & 1 & \frac{2s-1}{s^2+1} \end{array}\right)$$

$$sV(s) = \frac{-s^2-1-2s+1}{s^2+1} = -\frac{s^2+2s}{s^2+1}$$

$$V(s) = -\frac{s+2}{s^2+1} \Rightarrow \boxed{v(t) = -(\cos(t) + 2\sin(t))}$$

Note: Using the system you get both $x(t)$ & $v(t)$ in the same calculation

Q5 iii) $x'(t) = 3x(t) + v(t)$ $\quad x(0) = 1$

$v'(t) = x(t) + v(t)$ $\quad v(0) = -1$

Upon Laplace transforming we get

$$\begin{pmatrix} s\,X(s) - 1 \\ s\,V(s) + 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X(s) \\ V(s) \end{pmatrix}$$

$$\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} X(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} s-3 & -1 \\ -1 & s-1 \end{pmatrix} \begin{pmatrix} X(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The row reduction is just time consuming (not for test) so I skip the details to get

$$X(s) = \frac{s-2}{s^2-4s+2} \qquad V(s) = -\frac{s-4}{s^2-4s+2}$$

Maple is the perfect tool for inversion ~~&~~ gives

$$x(t) = e^{2t} \cosh(\sqrt{2}\,t)$$

$$v(t) = e^{2t}\left[\sqrt{2}\sinh(\sqrt{2}\,t) - \cosh(\sqrt{2}\,t)\right]$$

Q5 iii) cont'd The answer uses the hyperbolic sine and cosine

$$\sinh A = \frac{e^{A} - e^{-A}}{2} \; ; \; \cosh A = \frac{e^{A} + e^{-A}}{2}$$

$\sinh(0) = 0$ like $\sin(0)$

$\cosh(0) =$ [illegible] like $\cos(0)$

and $\cosh^2(A) - \sinh^2(A) = 1$ for all A

like the hyperbola $x^2 - y^2 = 1^2$

Q5 iv) Here the point is that changing the initial conditions only changes the right hand side and not the matrix itself:

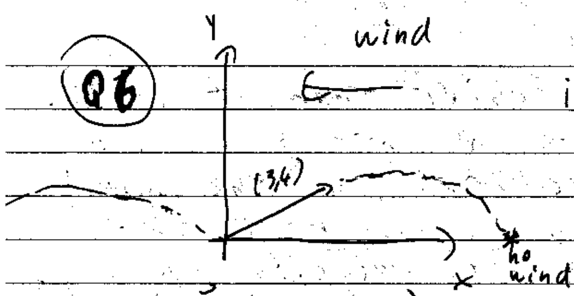
$$\begin{pmatrix} s-3 & -1 \\ -1 & s-1 \end{pmatrix} \begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$$

$$X(s) = \frac{s-11}{s^2-4s+2} \; ; \; V(s) = -\frac{10s-31}{s^2-4s+2}$$

$$x(t) = -\frac{1}{2} e^{2t} \left[9\sqrt{2}\sinh(\sqrt{2}t) - 2\cosh(\sqrt{2}t) \right]$$

$$v(t) = \frac{1}{2} e^{2t} \left[11\sqrt{2}\sinh(\sqrt{2}t) - 20\cosh(\sqrt{2}t) \right]$$

Q6



i) Newton's Law gives

$$m \frac{d v_2(t)}{dt} = -mg \; ; \; g = 10 \frac{m}{s^2}$$

$$\frac{dy}{dt} = v_2(t)$$

$$\frac{d v_1(t)}{dt} = 0$$

Let $\vec{v} = (v_1(t), v_2(t))$

$\therefore \; v_1(t) = v_1(0) = 3 \text{ m/s}$

Similarly, integrating gives $v_2(t) = 4 - 10t$

So that integrating again gives $y(t) = 4t - 5t^2$

and $y = 0$ when $t = 0$ or $4 - 5t = 0 \Rightarrow t = \frac{4}{5}$ s

Thus at $t = 0.8$ s the particle hits the ground again

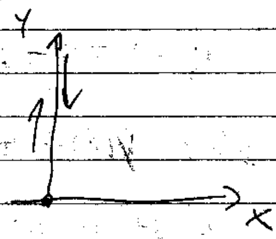
In 0.8 s the particle travels 2.4 m horizontally

ii) Adding a horizontal wind does nothing to the vertical velocity, so after 0.8 s the particle returns to the ground. However $v_1(t) = v_1(0) - 10 = -7 \text{ m/s}$ and the particle travels -5.6 m in the horizontal direction.

iii) This one is easier than it looks. Since there are no forces in the horizontal direction all the wind has to do is counteract the original horizontal velocity of 3 m/s

$\therefore V_w = -3$ m/s

and now the particle just moves up and down.



(Q7) plots are attached separately

i) $\cos^2(t) + \sin^2(t) = 1^2$ so we have a <u>circle</u>

ii) $\cos^2(2t) + \sin^2(2t) = 1^2$
a circle again but with a larger speed of travel (return to $(1,0)$ after $t = \pi$)

iii) $x(t)^2 + \left(\frac{y(t)}{\sqrt{3}}\right)^2 = 1^2$ an ellipse with a major axis along the y-axis

iv) same ellipse, but with a clockwise direction of travel

v) Again an ellipse but this time with the major axis along the x-axis

vi) Now the periodicity of $x(t)$ does not match $y(t)$. Because the two periods are a factor of two off we still expect a closed curve. Note $\vec{x} = (0,0)$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}$

so the curve crosses itself at the origin.

Q8 iv) Now $\frac{dx(t)}{dt} = \frac{1}{\sqrt{|1-t|}} = \begin{cases} \frac{1}{\sqrt{1-t}} & t<1 \\ \text{undefined} & t=1 \\ \frac{1}{\sqrt{t-1}} & t>1 \end{cases}$

we showed in part ii) that as $t \to 1^-$ $x(t) \to 2$ so if we take an instant after $t=1$, say $t=1.0001$ we could take

$$x(1.0001) = 2$$

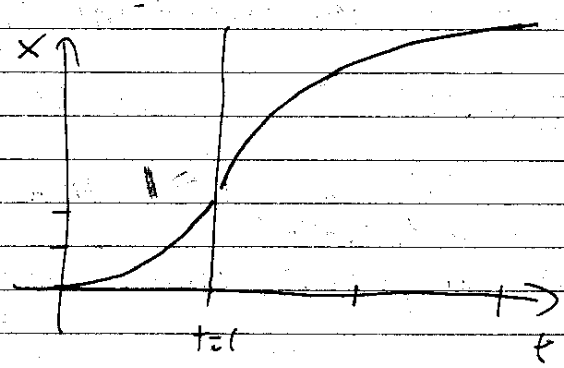
$$\frac{dx}{dt} = \frac{1}{\sqrt{t-1}} \quad \text{for } t \geq 1.0001$$

integrate to get

$$x(t) = 2\sqrt{t-1} + C$$

$x(1.0001) = 2$ gives $C = 1.98$

$$x(t) = 2\sqrt{t-1} + 1.98$$



Q8 i) This one is physically obvious, if two particles start in the same spot then the one moving faster will be ahead. To guarantee this mathematically we want

$$V_1(t) > V_2(t) \quad \text{for all } t > 0$$

ii) $x'(t) = \frac{1}{\sqrt{1-t}} = (1-t)^{-\frac{1}{2}}$

$\Rightarrow x(t) = -2(1-t)^{\frac{1}{2}} + C$ check: $x'(t) = -2 \cdot \frac{1}{2}(1-t)^{-\frac{1}{2}}(-1) = (1-t)^{-\frac{1}{2}}$ ✓

$x(0) = 0$ means

$-2 \cdot 1^{\frac{1}{2}} + C = 0$ or $C = 2$

$x(t) = 2 - 2(1-t)^{\frac{1}{2}}$

as $t \to 1$ from below $x(t) \to 2$

iii) If $x'(t) = (1-t)^{-1}$ then $x(t) = -\ln(1-t) + C$

$x(0) = 0 \Rightarrow -\ln(1) + C = 0$ or $C = 0$

$x(t) = -\ln(1-t)$ and as $t \to 1$ from below $x(t) \to +\infty$ i.e. undefinded so the same argument cannot work