Important: Assignment 9 covers the last five lectures on Taylor series, Taylor polynomials, and Taylor's Remainder Theorem. While it is not for submission, 10-15% of the final exam will be based on this material. Thus it is essential that you do as many of the problems as possible. Some have been suggested with an *. There will be a tutorial for Assignment 9 on Wednesday, March 29, from 3:30 - 5:30 in RCH 101. Solutions will be posted by April 5th.

Note: In this assignment, we use the notation $P_N(x)$ for the Taylor polynomial of order N. (In your text, and in Maple Lab 2, the notation $T_n(x)$ is used for order n.)

1. Find the specified Taylor polynomial $P_N(x)$ centred at $x = a$ for each of the given functions by evaluating $f(a)$, $f'(x)$, $f''(a), \ldots$ to determine the coefficients.

   *a) $f(x) = x \ln x$, $P_N(x)$, $a = 1$ *e) $f(x) = \int_e^x \frac{1}{\ln t} dt$, $P_3(x)$, $a = e$ (Use FTC1)

   *b) $f(x) = \arctan x$, $P_3(x)$, $a = 1$ f) $f(x) = \tan x$, $P_3(x)$, $a = \pi/4$

   c) $f(x) = \sin x$, $P_4(x)$, $a = \pi/2$ *g) $f(x) = \cos x$, $P_4(x)$, $a = \pi/3$

   d) $f(x) = \ln(1 - x)$, $P_N(x)$, $a = 0$ h) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $P_3(x)$, $a = 0$ (Use FTC1)

2. Find $P_3(x)$, the Taylor polynomial of degree 3 centred at $x = 0$, for each of the following functions by choosing p and x appropriately in the binomial expansion, and then truncating the series:

   a) $f(x) = \sqrt{1+x}$ b) $f(x) = (1+x)^{1/4}$

   *c) $f(x) = \frac{1}{\sqrt{4-x}}$ *d) $f(x) = \frac{1}{(2+x)^3}$

3. a) Use Taylor's Remainder Theorem to show that

$$\left| \sin x - \left(x - \frac{x^3}{6} \right) \right| \leq \frac{|x|^5}{120} \quad \text{for all} \quad x \in \mathbb{R}$$

   b) Find an approximation for $\sin(.3)$, and state an upper bound for the error.

   c) Following the method of Example 6 on page 133 of your Course Notes, use the result from a) to find an approximate value for $\int_0^{\pi/6} \sin(x^2) dx$, and derive an upper bound on the error. Compare this upper bound with the error bound predicted by the corollary of Alternating Series Test.

4. a) Use Taylor's Remainder Theorem to show that

$$\left| e^{-x} - \left(1 - x + \frac{x^2}{2} \right) \right| \leq \frac{x^3}{6} \quad \text{for all} \quad x \geq 0.$$

   b) Use the result from a) to find an approximate value for $\int_0^1 e^{-\left(\frac{t}{2}\right)^2} dt$ and find an upper bound on the error.

   HINT: Follow the same procedure as suggested in #3 above.

5. *a) State Taylor's Remainder Theorem with $n = 2$ and use it to show that

$$\left|\sqrt{1+x} - \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\right| \le \frac{x^3}{16} \quad \text{for all} \quad x \ge 0.$$

   b) Use the result in a) to approximate each quantity and find an upper bound on the error.

   (i) $\sqrt{1.02}$ (ii) $\sqrt{2}$ (Rewrite as $\frac{7}{5}\sqrt{1+\frac{1}{49}}$) *(iii) $\int_0^1 \sqrt{1+\frac{1}{2}t^4}\,dt$ (iv) $\int_0^\pi \sqrt{4+\sin t}\,dt$

*6. Suppose $P_2(x)$ centred at $x = a = 1$ for a certain function $f(x)$ is

$$P_2(x) = 2 + 4(x-1) + 3(x-1)^2.$$

Find the Taylor polynomial $T_2(x)$ centred at $x = 1$ for the function $g(x) = \dfrac{1}{f(x)}$.

7. a) Find a constant M such that

$$\left|\sqrt{1+x} - \left(1 + \frac{1}{2}x\right)\right| \le Mx^2 \quad \text{for} \quad x \in \left[-\frac{1}{2}, 0\right].$$

   b) Find $P_3(x)$ for the function $f(x) = \ln(1+x)$, centred at $x = a = 0$. If we wish to have error

$$|f(x) - P_3(x)| \le 2.5 \times 10^5 \quad \text{for} \quad x \in [-d \le x \le d],$$

   what is the largest possible value for d? (You will need Maple to solve the equation for d.)

   *c) Find a value of N which guarantees that

$$\left|e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}\right)\right| \le 10^{-4} \quad \text{for} \quad x \in [-1, 1].$$

8. a) Show that if $f(x)$ is an odd function, then any Taylor polynomial $P_N(x)$ centred at $x = a = 0$ will contain only odd powers of x.

   b) Prove a similar result for even $f(x)$.

9. Taylor's Remainder Theorem can be used to evaluate limits. For example, we know that $e^x = 1+x+kx^2$, where $k = \frac{e^c}{2!}$ for some c between 0 and x (TRT for $a = 0$, $n = 1$). Thus

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{x\to 0} \frac{1 + x + kx^2 - 1}{x} \\ &= \lim_{x\to 0} (1 + kx) \\ &= 1.\end{aligned}$$

Use a similar method to find each limit.

*a) $\displaystyle\lim_{x\to 0} \frac{1-e^{-x^2}}{1-\cos x}$ b) $\displaystyle\lim_{x\to 0} \frac{2\sqrt{1+x} - 2 - x}{x^2}$ *c) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$

10. *a) You are about to calculate $\sin(36°)$ when the batteries die on your calculator. You roommate has an old calculator, but it has no 'sin' key. Unperturbed, she enters 3.1415926, divides by 5, enters the result in memory (called 'x' hereafter). Then she calculates $x\left(1-\dfrac{x^2}{6}\right)$ and uses it for the value of $\sin(36°)$. Explain what she was doing in terms of Taylor polynomials, find the answer she got, and give an upper bound on the error.

b) The electrical potential energy V at a point P due to the charge on a disc of radius a and constant charge density σ is

$$V(P) = 2\pi\sigma\left(\sqrt{R^2+a^2} - R\right),$$

where R is the distance from p to the disc. Use the substitution $a = Rx$ and the first two terms of the Maclaurin series for $\sqrt{1+x^2}$ to show that for large R, (i.e. for $x = a/R$ near 0), $V(P) \approx \pi a^2\sigma/R$.

c) Taylor series approximations can be used to find approximate solutions to equations. A nice example of this is the equation

$$\sin x + b(1 + \cos^2 x + \cos x) = 0,$$

where b is a very small positive constant. (This equation arose in Einstein's calculations predicting the bending of light by the gravitational field of the sun.)

i) Explain how you know there is a solution near $x = 0$.

ii) Expand the equation about $x = 0$, using only the *linear* terms in x (i.e. disregard terms of order x^2 or higher), and solve for x (in terms of b).