MATH 136 **SOLUTIONS to Assignment 7 Fall/05**

Due: Thursday Nov. 17/05
(Grade is out of 32.)

Contents

1 Linear Independent Sets of Vectors

9 marks

Let V be the vector space of all continuous real valued functions defined on the interval $[0, \pi]$. Consider the following subsets of V. Which of the subsets are linearly independent and why?

1.
$$S_1 = \{\sin(t), \cos(t)\}, \quad \text{(two functions)}$$

2.
$$S_2 = \left\{\sin^4(t), \cos^4(t), \sin^2(t)\cos^2(t)\right\}, \quad \text{(three functions)}$$

3.
$$S_3 = \left\{\sin^4(t), \cos^4(t), \sin^2(t)\cos^2(t), 3.1\right\}, \quad \text{(four functions)}$$

Note: $f(t) = \sin^2(t)\cos^2(t)$ is a function of t and $g(t) = 3.1$ is also a function of t, i.e. the latter is the constant function that takes the value 3.1 for all t.

SOLUTIONS

We first define the operators '+','-' and scalar multiplication '$\cdot$' on the sets of real valued functions:

(a) $f_1 + f_2$: $[0, \pi] \longrightarrow R$
$\forall t \in [0, \pi], \quad (f_1 + f_2)(t) = f_1(t) + f_2(t)$

(b) $-f_1$: $[0, \pi] \longrightarrow R$
$\forall t \in [0, \pi], \quad (-f_1)(t) = -(f_1(t))$

(c) cf_1: $[0, \pi] \longrightarrow R$
$\forall t \in [0, \pi], \forall c \in R, \quad (cf_1)(t) = c(f_1(t))$

With these definitions, V is a vector space over the reals.
Thus:

Solution of Part 1: S_1 is linearly independent, i.e. suppose

$$\exists c_1, c_2 \in R, (c_1 \sin)(t) + (c_2 \cos)(t) = 0\,(t),$$

where $0\,(t)$ denotes the identically zero function. Notice in the above equality, we are using real valued continuous functions rather than numbers. The above equality of functions is equivalent to the following:

$$\forall t \in [0, \pi], \quad c_1 \sin(t) + c_2 \cos(t) = 0\,(t).$$

Take $t = 0$, we get the equation $c_1 0 + c_2 1 = 0$ and take $t = \frac{\pi}{2}$, we get the equation $c_1 1 + c_2 0 = 0$. The two equations have coefficient matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This implies $c_1 = 0, c_2 = 0$. Therefore, the only linear combination to yield the zero function is the trivial linear combination, i.e. and $\cos, \sin$ are two linearly independent vectors (functions) in V.

Solution of Part 2: S_2 is linearly independent, i.e. suppose $\exists c_1, c_2, c_3 \in R$, such that

$$(c_1 \sin^4)(t) + (c_2 \cos^4)(t) + c_3(\cos^2 \sin^2)(t) = 0\,(t), \quad \forall t.$$

Then, plugging in the value $t = 0$, we get $c_2 = 0$. Now if we choose $t = \frac{\pi}{2}$, we have $c_1 = 0$. The equality has reduced to

$$\forall t \in [0, \pi], 0 * \sin^4(t) + 0 * \cos^4(t) + c_3 * \sin^2(t) \cos^2(t) = 0\,(t).$$

This is just

$$\forall t \in [0, \pi], c_3 \sin^2(t) \cos^2(t) = 0.$$

Choose $t = \frac{\pi}{4}$. Then $\sin^2(t) \cos^2(t) = 0.25 \neq 0$. Thus $c_3 = 0$. As above, this means we only have the trivial linear combination yielding the zero function. Therefore, the three vectors (functions) $\sin^4(t), \cos^4(t), (\cos^2 \sin^2)(t)$ are linearly independent.

Solution of Part 3: S_3 is linearly dependent, i.e. take $c_1 = 1, c_2 = 1, c_3 = 2, c_4 = -\frac{1}{3.1}$. (We are using the fact that $(\cos^2 t + \sin^2 t)^2 = 1^2 = 1$.) We have $\forall t \in [0, \pi]$

$$\begin{aligned} c_1 \sin^4(t) + c_2 \cos^4(t) + c_3 \cos^2(t) \sin^2(t) + c_4(3.1) &= (\sin^2(t) + \cos^2(t))^2 - 1 \\ &= = 1 - 1 = 0\,. \end{aligned}$$

Therefore, we have a nontrivial ilnear combination which yields the zero function, i.e. the set is linearly dependent.

2 A Rotation in the Plane

12 marks

Suppose that the vector in the plane $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is given. Define the transformation T on v to be the clockwise rotation in the plane through an angle $\theta = 45$ degrees, i.e. $T(v)$ is the vector in the plane obtained by rotating v clockwise 45 degrees. Similarly, define the transformation S on v to be the counter-clockwise rotation in the plane through an angle $\theta = 60$ degrees, i.e. $S(v)$ is the vector in the plane obtained by rotating v counter-clockwise 60 degrees.

1. Show that T (and so also S) is a linear transformation and find the matrix representations T_A, T_S of T and S, respectively.
2. What is $W(v) = S(T(v))$? Find a simpler description of the product $W = ST$; and find a matrix representation T_W of W.
3. Confirm that $T_W = T_S T_T$.

SOLUTIONS

1. If we rotate a vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ clockwise with angle θ, then we can see geometrically that this is just left multipication by the matrix:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

So $\forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$,

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cos(\frac{\pi}{4}) & sin(\frac{\pi}{4}) \\ -sin(\frac{\pi}{4}) & cos(\frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Similarly, counterclockwise rotation is just clockwise rotation with a negative angle. Therefore,

$$S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cos(-\frac{\pi}{3}) & sin(-\frac{\pi}{3}) \\ -sin(-\frac{\pi}{3}) & cos(-\frac{\pi}{3}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Because S and T could be represented as matrix multiplication with the matrices $A_T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$, $A_S = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, respectively, both S, T are linear transformations.

2. $W(v) = S(T(v))$ is firstly rotate v clockwise by 45 and counter-clockwise by 60 is equivalent to rotate v counter-clockwise by -15, so the matrix representing W is

$$T_w = \begin{pmatrix} cos(\frac{\pi}{12}) & sin(-\frac{\pi}{12}) \\ -sin(-\frac{\pi}{12}) & cos(\frac{\pi}{12}) \end{pmatrix}$$

3. We just need to check $T_s * T_a = T_w$:

$$\begin{pmatrix} cos(\frac{\pi}{3}) & sin(-\frac{\pi}{3}) \\ -sin(-\frac{\pi}{3}) & cos(\frac{\pi}{3}) \end{pmatrix} * \begin{pmatrix} cos(\frac{\pi}{4}) & sin(\frac{\pi}{4}) \\ -sin(\frac{\pi}{4}) & cos(\frac{\pi}{4}) \end{pmatrix}$$

$$= \begin{pmatrix} cos(\frac{\pi}{3})cos(\frac{\pi}{4}) + sin(-\frac{\pi}{3})sin(-\frac{\pi}{4}) & cos(\frac{\pi}{3})sin(\frac{\pi}{4}) + sin(-\frac{\pi}{3})cos(\frac{\pi}{4}) \\ sin(\frac{\pi}{3})cos(\frac{\pi}{4}) + cos(\frac{\pi}{3})sin(-\frac{\pi}{4}) & sin(\frac{\pi}{3})sin(\frac{\pi}{4}) + cos(\frac{\pi}{3})cos(\frac{\pi}{4}) \end{pmatrix}$$

$$= \begin{pmatrix} cos(\frac{\pi}{3} - \frac{\pi}{4}) & sin(\frac{\pi}{4} - \frac{\pi}{3}) \\ sin(\frac{\pi}{3} - \frac{\pi}{4}) & cos(\frac{\pi}{3} - \frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} cos(\frac{\pi}{12}) & -sin(\frac{\pi}{12}) \\ sin(\frac{\pi}{12}) & cos(\frac{\pi}{12}) \end{pmatrix}$$

which is just T_w.

3 Page 235, Problem 40

4 marks

Since $w \in \text{span}\{v_1, v_2\}$, we know w is the linear combination of v_1 and v_2, so $\exists c_1, c_2 \in R, w = c_1v_1 + c_2v_2$, similarly, since $w \in \text{span}\{v_3, v_4\}$, $\exists c_3, c_4 \in R,\ \ w = c_3v_3 + c_4v_4$. Thus we get the equality:

$$c_1v_1 + c_2v_2 = w = c_3v_3 + c_4v_4.$$

So

$$c_1v_1 + c_2v_2 + c_3 * (-v_3) + c_4(-v_4) = 0$$

Convert this into matrix form:

$$\begin{pmatrix} v_1 & v_2 & -v_3 & -v_4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$$

we program with matlab, the program is:

```
A=[ 5 1 -2 0 0;
3 3 1 12 0;
 8 4 -5 28 0];
```

```
A(1,:)=A(1,:)/A(1,1);
A(2,:)=A(2,:)-A(1,:)*A(2,1);
A(3,:)=A(3,:)-A(3,1)*A(1,:);
A(2,:)=A(2,:)/A(2,2);
A(3,:)=A(3,:)-A(3,2)*A(2,:);
A(3,:)=A(3,:)/A(3,3);
A(1,:)=A(1,:)-A(1,3)*A(3,:);
A(2,:)=A(2,:)-A(2,3)*A(3,:);
A(1,:)=A(1,:)-A(1,2)*A(2,:)
```

and get the following result:

```
A =

    1.0000    0.2000   -0.4000         0         0
         0    2.4000    2.2000   12.0000         0
    8.0000    4.0000   -5.0000   28.0000         0


A =

    1.0000    0.2000   -0.4000         0         0
         0    2.4000    2.2000   12.0000         0
         0    2.4000   -1.8000   28.0000         0


A =

    1.0000    0.2000   -0.4000         0         0
         0    1.0000    0.9167    5.0000         0
         0    2.4000   -1.8000   28.0000         0


A =

    1.0000    0.2000   -0.4000         0         0
         0    1.0000    0.9167    5.0000         0
         0         0   -4.0000   16.0000         0


A =

    1.0000    0.2000   -0.4000         0         0
         0    1.0000    0.9167    5.0000         0
         0         0    1.0000   -4.0000         0
```

```
A =

    1.0000    0.2000         0   -1.6000         0
         0    1.0000    0.9167    5.0000         0
         0         0    1.0000   -4.0000         0


A =

    1.0000    0.2000         0   -1.6000         0
         0    1.0000         0    8.6667         0
         0         0    1.0000   -4.0000         0


A =

    1.0000         0         0   -3.3333         0
         0    1.0000         0    8.6667         0
         0         0    1.0000   -4.0000         0
```

Therefore, $t\begin{pmatrix} \frac{10}{3} \\ -\frac{26}{3} \\ 4 \\ 1 \end{pmatrix}, t \in R$ is the general solution. Thus w could be represented as $c_1v_1 + c_2v_2$, which equals$\begin{pmatrix} v_1 & v_2 \end{pmatrix} * \begin{pmatrix} -\frac{10}{3}t \\ \frac{26}{3}t \end{pmatrix} = t\begin{pmatrix} -8 \\ 16 \\ 8 \end{pmatrix}, t \in R$.

4 Page 243, Problem 4

3 marks

Firstly we convert the matrix $[v_1, v_2, v_3]$ into echelon form. and they are linearly independent if and only if every column has a pivot; and they span R^3 if and only if every row has a pivot. the matlab function is as following:

```
A=[2 1 -7;-2 -3 5;1 2 4]; A(2,:)=A(2,:)-A(1,:)*A(2,1)/A(1,1)
A(3,:)=A(3,:)-A(3,1)*A(1,:)/A(1,1)
A(3,:)=A(3,:)-A(3,2)*A(2,:)/A(2,2)
```

The result is as following:

```
A =

     2     1    -7
     0    -2    -2
     1     2     4
```

```
A =

    2.0000    1.0000   -7.0000
         0   -2.0000   -2.0000
         0    1.5000    7.5000


A =

     2     1    -7
     0    -2    -2
     0     0     6
```

So they are linearly independent and span R^3, so the three vectors are a basis of R^3.

5 Page 243, Problem 10

4 marks

Firstly reduce A to reduced echelon form. The matlab function is as

```
A=[1 0 -5 1 4;-2 1 6 -2 -2;0 2 -8 1 9]; A(1,:)=A(1,:)/A(1,1);
A(2,:)=A(2,:)-A(1,:)*A(2,1)
A(3,:)=A(3,:)-A(3,1)*A(1,:)
A(2,:)=A(2,:)/A(2,2)
A(3,:)=A(3,:)-A(3,2)*A(2,:)
A(3,:)=A(3,:)/A(3,3)
A(1,:)=A(1,:)-A(1,3)*A(3,:)
A(2,:)=A(2,:)-A(2,3)*A(3,:)
A(1,:)=A(1,:)-A(1,2)*A(2,:)
```

The result is as following:

```
A =

     1     0    -5     1     4
     0     1    -4     0     6
     0     2    -8     1     9


A =

     1     0    -5     1     4
     0     1    -4     0     6
     0     2    -8     1     9
```

```
A =

     1     0    -5     1     4
     0     1    -4     0     6
     0     2    -8     1     9


A =

     1     0    -5     1     4
     0     1    -4     0     6
     0     0     0     1    -3

Warning: Divide by zero.
```

We complete the reduction process and get the coefficients matrix as: $\begin{pmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$
and we deduce the general solution, i.e. the Null space of A is:

$$Nul(A) = \{t \begin{pmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{pmatrix} | t, s \in R\}$$

and $\begin{pmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{pmatrix}$ are linearly independent, so they are a basis of $Nul(A)$.