# Math 138 Physics Based Section Assignment 3

**Q1** Consider the very simplest Simple Harmonic Oscillator equation

$$\frac{d^2x}{dt^2} + x = 0$$

i) Use the Laplace transform to solve the initial value problem with $x(0) = 1$ and $x'(0) = 0$

Recall that

$$\mathcal{L}(x''(t)) = s^2X(s) - sx(0) - x'(0) = s^2X(s) - s$$

where I use the initial conditions in the second step. The governing equation after Laplace transforming reads:

$$s^2X(s) - s + X(s) = 0.$$

Isolating for $X(s)$ gives

$$X(s) = \frac{s}{s^2+1}$$

and from the notes we get

$$x(t) = \cos(t)$$

ii) using part i) show that the initial value problem $x(0) = x_0$ and $x'(0) = 0$ has the solution

$$x(t) = x_0 \cos(t))$$

The only change from part i) is that $x(0) = x_0$ so that

$$s^2X(s) - sx_0 + X(s) = 0.$$

or

$$X(s) = x_0 \frac{s}{s^2+1}$$

and

$$x(t) = x_0 \cos(t)$$

iii) Show, using Laplace transforms, that if we change the governing equation to

$$\frac{d^2x}{dt^2} + 4x = 0$$

then the solution to the initial value problem is $x(0) = 1$, $x'(0) = 0$ is $x(t) = \cos(2t)$.

Because the Laplace transform is linear we have, upon transforming the governing equation

$$s^2X(s) - s + 4X(s) = 0$$

or

$$X(s) = \frac{s}{s^2 + 2^2}$$

which immediately gives us

$$x(t) = \cos(2t)$$

iv) Use Laplace transforms to solve the initial value problem $x(0) = 0$ and $x'(0) = 1$. For the governing equation in part iii)
Upon Laplace transforming the governing equation reads:

$$s^2 X(s) - 1 + 4X(s) = 0$$

or

$$X(s) = \frac{1}{s^2 + 2^2} = \frac{1}{2} \frac{2}{s^2 + 2^2}$$

using the notes and the linearity of the Laplace transform we get:

$$x(t) = \frac{1}{2} \sin(2t)$$

v) Explain the factor of 1/2 you found in part iv)
By Chain Rule

$$\frac{d}{dt} \sin(2t) = 2\cos(2t)$$

thus to get $x'(0) = 1$ we need to divide out by the two that comes from taking the derivative.

vi) If

$$x'' + x = \sin(t)$$

solve the initial value problem with $x(0) = 0$ and $x'(0) = 1$.
Taking Laplace transforms and using the notes for the transform of sine gives

$$s^2 X(s) - sx(0) - x'(0) + X(s) = \frac{1}{s^2 + 1}$$

or upon isolating for $X(s)$

$$X(s) = \frac{1}{s^2 + 1} + \frac{1}{(s^2 + 1)^2}.$$

The first term is easy enough (it is the transform of sine) but the second is trouble. It doesn't look like anything in tables AND partial fractions don't work if you try them. However, if you recall that in class we showed that when the function on the right hand side looks like a solution then a multiple of $t$ often popped up in the solution. Indeed checking the table of transforms I have posted we find that (after a bit of algebra)

$$x(t) = \frac{3}{2} \sin(t) - \frac{1}{2} t \cos(t)$$

vii) In the previous solution describe the dominant behaviour for large $t$.

Since sinusoids are periodic the large time behaviour is determined by the "envelope" function. for the first term the envelope is a constant (namely three halves) while for the second term it is $-t/2$. Thus for large $t$ the solution still oscillates, but the maximum amplitude grows as $t$. This is commonly known as the phenomenon of **resonance**.

**(Q2)** Consider the damped oscillator equation

$$x''(t) + bx'(t) + x(t) = 0$$

where $b > 0$ ON PHYSICAL GROUNDS.
i) Let $x(t) = \exp(at)$ and show that $a$ satisfies the quadratic equation

$$a^2 + ba + 1 = 0$$

By the definition of the exponential

$$\frac{d}{dt}x(t) = ax(t)$$

and

$$\frac{d^2}{dt^2}x(t) = a^2x(t)$$

so that the answer follows by simple substitution.
ii) If $b^2 > 4$ show that $a$ has two solutions.
The solution of the quadratic that determines $a$ are given by the quadratic formula. This means that the NATURE of the solutions is given by the discriminant:

$$D = b^2 - 4$$

so that two real solutions are possible when $b^2 > 4$.
iii) For what values of $b$ can we get a positive solution for $a$?

The quadratic formula yields

$$a = \frac{-b \pm \sqrt{b^2 - 4}}{2}$$

so to guarantee $a > 0$ we need $-b + \sqrt{b^2 - 4} > 0$ or $\sqrt{b^2 - 4} > b$. We know $b > 0$ so that $-b < 0$ and thus we need to take the root with plus in it. Square both sides to get $b^2 - 4 > b^2$ which is NEVER true. This is quite a neat result.

iv) Interpret the result from part iii).
As long as $b > 0$ we have that as $t \longrightarrow \infty$ $x(t) \longrightarrow 0$ NO MATTER WHAT $b$ IS.

v) If $b = \sqrt{3}$ find $a$

From the quadratic formula we have

$$b = -\frac{\sqrt{3}}{2} \pm \frac{\sqrt{3-4}}{2} = -\frac{\sqrt{3}}{2} \pm i\frac{1}{2}$$

vi) Show that in this case $x(t) = \exp\left(-\sqrt{3}t/2\right)\sin\left(\frac{1}{2}t\right)$ is a possible solution.

This one just follows by the Product Rule and messy algebra. Apart from tests, Maple make short work of a question like this. On a test I would make the algebra simpler.

**Q3**i) Without solving find one interesting fact about the solution to the initial value problem

$$x'(t) = \tan(t)$$

where $x(0) = 2$.

Recall that the right hand-side tells us the RATE OF CHANGE. Since we are told that the rate of change is the tangent function we know that as $t \longrightarrow \pi/2$ the rate of change grows without bound. Thus whatever the solution to the initial value problem is, it CANNOT be valid for inputs $t \geq \pi/2$.

ii) Without solving, show that the solution to the initial value problem

$$x'(t) = x^2(10 - x)$$

where $x(0) = 2$ remains bounded for all $t > 0$

We first find the places where $x'(t) = 0$, or the *fixed points*. This occurs when $x^2(10-x) = 0$ so when $x = 0$ or $x = 10$. The solution of the initial value problem that starts at $x = 2$ stays in the interval $0 \leq x(t) \leq 10$ for all time.

iii) For the problem in part ii) what happens for large times?

We already know that $x(t)$ is trapped between 0 and 10. The only question is whether it increases toward 10 or decreases toward 0. Since $x^2 > 0$ and $x < 10$ initially we have $x'(t) > 0$ and so $x(t)$ INCREASES and thus must tend to 10 for large times.

iv) If $x'(t) < 0$ for all $t$ does $x \longrightarrow -\infty$ as $t \longrightarrow \infty$?

No, just think of the last problem and change it a bit to have the DE read:

$$x'(t) = x^2(x - 10).$$

If $x(0) = 2$ then $x'(t) < 0$ for all time but $x \longrightarrow 0$ as $t \longrightarrow \infty$.

v) If $x \longrightarrow -\infty$ as $t \longrightarrow \infty$ does $x'(t) < 0$ for all $t$?

This one is a bit deceitful. After all we can have $x(t)$ do whatever we want for a long, long time, before it eventually decreases to minus infinity. If you want a simple example consider

$$x'(t) = 1 - t^2$$

which has $x'(t) > 0$ for $0 < t < 1$ but still tends to minus infinity for large times.

**Q4** Consider a cylindrical tube with a radius of 2 meters. The tubing is rusting and thus the concentration of iron in the tube is given by

$$C(r) = C_0(0.5 + r/2)$$

i) What are the appropriate units for the reference concentration $C_0$? Why write the formula the way we did?
This isn't chemistry class, so I must be after a geometric idea. Here I just want you to notice that concentration must be the total amount of stuff per unit volume, or moles per meter cubed. As for why write it the way we did, notice that we could rewrite things as

$$\frac{C(r)}{C_0} = 0.5 + \frac{r}{R_{pipe}}$$

so that on both the left and right the dimensions cancel.

ii) If we now make things easy and set $C(r) = 0.5 + r/2$ use the cylindrical shells technique to find the total amount of iron in a 3 meter section of pipe.

Here we note that nothing actually varies as a function of along pipe distance, so we just find the total concentration in one circular section of the pipe and multiply by the total height of three meters. If we let the amount of iron in a band of radius $r$ be denoted by $Fe(r)$ then

$$Fe(r) = \text{Area of band} \times C(r) = 2\pi r \Delta r C(r)$$

and the total amount of iron in a circular cross-section is a sum over the shells:

$$Fe_{area} = \sum Fe(r).$$

as the boxes get thin ($\Delta r \longrightarrow 0$) we must have

$$Fe_{area} = \int_0^{R_{pipe}} 2\pi r C(r) dr = 2\pi \int_0^2 (0.5r + r^2/2) dr = 2\pi \times \frac{7}{3}$$

where I skipped some of the simple integration steps at the end. This is just the total amount of stuff in ONE circular cross-section, thus we multiply by total height of three meters to get

$$Fe_{total} = 14\pi.$$

iii) What is the average concentration in the pipe?

Here again you need to consider in what direction things are changing. You should conclude that again it is only the $r$ direction that matters. However we need to be careful about how to do the "adding". This is because a donut of fixed width $\Delta r$ has much more area when $r$ is bigger. Moreover the average should be the total "stuff" divided by the total area:

$$< C >= \frac{1}{\pi R_{pipe}^2} \int_{r=0}^{r=2} 2\pi r C(r) dr = \frac{7}{6}$$

If you look at part ii) carefully you will note that the result for the total was just the average times area of the pipe times the total length of pipe:

$$\frac{7}{6} \pi R_{pipe}^2 \times 3 = 14\pi$$

which, when you think about it a bit, is just what the definition of average should be.

iv) If $C(r) = C_0 + \sin(\pi r)$ explain WITHOUT calculating why the average of $C(r)$ is not merely $C_0$. Could you change $C(r)$ to get the right result?

It is certainly true that $C(r)$ is periodic as a function of $r$, however we know from the previous part that it is not just $C(r)$ that matters but also the area of each "donut" shaped region, and these increase as $r$ increases and change the weighting:

$$< C >= \frac{1}{\pi 2^2} \int_0^2 2\pi r C(r) dr = C_0 + \frac{1}{2} \int_0^2 r C(r) dr$$

from which we see that

$$C(r) = C_0 + \frac{\sin(\pi r)}{r}$$

will work.

**Q5**i) Consider the equation that governs the decaying exponential $y(t) = C_0 \exp(-2t)$:

$$y'(t) + 2y = 0$$

Use the trick for separable equations to find the solution and thus explain why the arbitrary constant is multiplied and not added.
We write

$$\frac{dy}{dt} + 2y = 0$$

or

$$\frac{dy}{y} = -2dt$$

which we integrate to get

$$\ln(y) = -2t + C.$$

Next exponentiate both sides to get:

$$\exp(\ln(y(t))) = \exp(-2t + C) = \exp(C)\exp(-2t)$$

The left hand side simplifies and $C$ is arbitrary so that in the end we get

$$y(t) = C_0 \exp(-2t)$$

as desired. The arbitrary constant multiplies because of the property of exponentials that turns multiplication into the addition of exponents.

ii) Solve the separable DE

$$y'(t) - \frac{y}{t} = 0$$

and sketch some solution curves.
Write

$$\frac{dy}{y} = \frac{dt}{t}$$

which can be integrated to read

$$\ln(y) = \ln(t) + C.$$

Exponentiate both sides to get

$$y(t) = C_0 t$$

so that solutions are straight lines which ALL pass through the origin and whose slope is given by the arbitrary constant (or physically the initial conditions, though they are somewhat strange in this case).

iii) Does the initial value problem $x(0) = 3$ for the previous equation have a solution?

No, we showed in part ii) that all solutions must have $x(0) = 0$ and thus $x(0) = 3$ is impossible to satisfy.

**(Q6)**i) Consider a skydiver falling under the influence of gravity and a drag force whose magnitude is given by $kv(t)$ and which acts AGAINST the direction of propagation.
i) Formulate the problem using Newton's Law for $v(t)$ NOT $x(t)$.
We orient the axis to point downwards and since the skydiver will fall in the positive direction we must have the drag force acting in the negative direction. The usual assumptions apply and then Newton's Second law gives us:

$$ma = mg - kv$$

which we can rewrite as

$$m\frac{dv}{dt} = mg - kv$$

ii) Show that if there is no acceleration then $v(t) = v_{terminal} > 0$.
No acceleration means $v'(t) = 0$ and so Newton's law reads

$$0 = mg - kv(t)$$

or

$$v(t) = \frac{mg}{k}.$$

This is called the "terminal velocity"
iii) Discuss how terminal velocity depends on the various physical parameters in the problem. From part ii) we find that heavier objects have a larger terminal velocity, that the velocity decreases as the drag coefficient increases and that a larger $g$ increases the terminal velocity.
iv) Solve the initial value problem with $v(0) = 0$ and show that $v(t)$ tends to the terminal velocity.
You can do this one with Laplace transforms, though I thought you should see a different way. Rewrite as

$$\frac{mdv}{mg - kv} = dt$$

or

$$\frac{dv}{g - \frac{kv}{m}} = dt$$

where we note that

$$\frac{d}{dv} \ln (g - kv/m) = \frac{1}{g - kv/m}(-k/m)$$

so that integrating the DE gives

$$\frac{-m}{k} \ln (g - kv/m) = t + C.$$

Multiply by $-m/k$ to get

$$\ln (g - kv/m) = -\frac{k}{m}t + D$$

where I have just renamed that arbitrary constant. Now exponentiate both sides and get $g - kv(t)/m = \exp(D)\exp(-kt/m)$ which we can rearrange to get

$$v(t) = \frac{mg}{k} - \frac{m}{k}\exp(D)\exp(-kt/m).$$

Now as $t \longrightarrow \infty$ the decaying exponential goes to zero and we are left with the first, constant term, which gives the terminal velocity.