Math 138 Assignment #6

Q1) i) Consider $a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

because $0 < \frac{1}{2} < 1$ we expect $a_n \to 0$ as $n \to \infty$. To prove this choose $\varepsilon > 0$ and notice if $a_n \to 0$ we need N so that $n > N$ guarantees

$$\left|\left(\tfrac{1}{2}\right)^n\right| < \varepsilon$$

because every thing is positive we remove the absolute value & get

$$\left(\tfrac{1}{2}\right)^n < \varepsilon \quad \text{or} \quad \tfrac{1}{2} < \varepsilon^{\frac{1}{n}}$$

This does not help much but we could logarithms:

$$\ln\left(\left(\tfrac{1}{2}\right)^n\right) < \ln(\varepsilon)$$

or $$n \ln\left(\tfrac{1}{2}\right) < \ln(\varepsilon)$$

Notice that for small $\varepsilon > 0$ $\ln(\varepsilon) < 0$

Now notice $\ln\left(\frac{1}{2}\right) < 0$ in fact $\ln\left(\frac{1}{2}\right) = -\ln(2)$

So $$n > \frac{-\ln(\varepsilon)}{\ln(2)}$$

The number on the right hand side is fixed so n will eventually get bigger

Q1 i) cont'd | Choose N to be the first positive integer bigger than

$$\frac{-\ln \varepsilon}{\ln 2}$$

Then $n > N$ guarantees

$$|a_n - 0| < \varepsilon$$

Q1 ii) Just like part i) except now we need

$$\left(\frac{1}{3}\right)^n < \varepsilon \quad \text{or} \quad n > \frac{-\ln \varepsilon}{\ln 3}$$

Take N as the first positive integer bigger than $-\dfrac{\ln \varepsilon}{\ln 2}$

iii) $a_n = \sin(n) \exp\{-n\}$

Notice $|a_n| = |\sin(n) \exp\{-n\}|$

$$\leq |\sin(n)| \, |\exp\{-n\}| \leq \exp\{-n\}$$

Here I used $|\sin(n)| \leq 1$ for all n and that $\exp\{-n\} > 0$ for all n

Q1 iii) Now $\exp\{-n\} \to 0$ as $n \to \infty$

so given $\varepsilon > 0$ we need to find N so that

$$|\sin(n)\exp\{-n\}| < \varepsilon$$

for all $n > N$.

That looks hard why not <u>instead</u> find N so that

$$\exp\{-n\} < \varepsilon$$

for all $n > N$? If we can do that then

$$|\sin(n)\exp\{-n\}| \leq \exp\{-n\}$$

will ensure our result.

Ok so to get N we want

$$\exp\{-n\} < \varepsilon$$

$$\text{or} \quad \ln(\exp\{-n\}) < \ln \varepsilon$$

$$-n < \ln \varepsilon$$

$$\text{or} \quad n > -\ln \varepsilon$$

We thus choose N to be the first positive integer larger than $-\ln \varepsilon$.

iv) $a_n = \sin(n)$ oscillates without repeating because sine has a period of 2π, an irrational number. To prove no limit exists we need to evaluate $\sin(n)$, which is not very easy to do.

Q2 i) $\frac{1}{n} \to 0$ as $n \to \infty$

so $\exp\{\frac{1}{n}\} \to \exp\{0\} = 1$

ii) $\arctan(m) \to \frac{\pi}{2}$ as $m \to \infty$

and n^5 grows without bound as $n \to \infty$, thus $\arctan(n^5) \to \frac{\pi}{2}$ as $n \to \infty$

iii) $a_n = \sin(\frac{n\pi}{5})$

sine has a period of 2π.

Clearly $n=1$ and $n=11$ give

$\sin(\frac{\pi}{5})$ and $\sin(\frac{10\pi}{5} + \frac{\pi}{5}) = \sin(2\pi + \frac{\pi}{5})$
$= \sin(\frac{\pi}{5})$

So a_n gives only 10 different values that repeat over and over again. a_n is thus <u>bounded</u>. It is not, however, <u>monotonic</u> so that the fact that a_n repeats forever and <u>does not</u> have a limit is not a violation of the <u>bounded monotonic sequence theorem</u>.

iv) $\exp\{-3n\} \to 0$ as $n \to \infty$

Thus $a_n = 5 - \exp\{-3n\} \to 5$ as $n \to \infty$

Q3 i) $a_n = \frac{1}{n}$ $b_n = -1$ for all n

$a_n \to 0$ $b_n \to -1$

ii) $a_n = 1$ $b_n = 0$ for all n

$a_n \to 1$ $|b_n| \le |a_n|$ but $b_n \to 0$

iii) Let $b_n = (-1)^n = (-1, 1, -1, 1, \ldots$

$|b_n| = 1$ for all n so b_n is

<u>bounded</u>. Clearly b_n has no

limit. Let $a_n = 0$ for all n

then $|a_n| \le |b_n|$ for all n

but $a_n \to 0$ as $n \to \infty$

iv) Let $a_n = n$ $b_n = -n$

a_n grows without bound

b_n gets more and more negative without bound

$c_n = a_n + b_n = n - n = 0$ so $c_n \to 0$ as $n \to \infty$

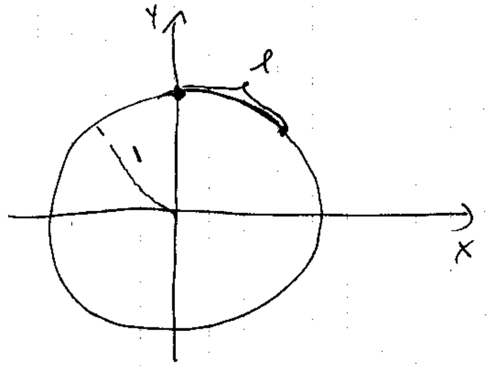
v) let $a_n = b_n = (-1)^n$. Clearly both have

no limit. Now $c_n = a_n b_n = (-1)^n (-1)^n$

$= (-1)^{2n} = [(-1)^2]^n = 1^n = 1$

$\therefore$ $c_n \to 1$ as $n \to \infty$

(Q4) You can play with this one for a while. The key thing to notice is that the distance around the circle is $2\pi r = 2\pi$ for the unit circle.



This is an <u>irrational</u> number, so if ℓ = rational number $\cdot\, 2\pi$, say $\ell = \frac{p}{q} 2\pi$ then eventually you will return to the same spot, otherwise you will not.
For $\ell = \frac{p}{q} 2\pi$ you return after q steps.

To show that an ℓ value that is <u>not</u> a rational multiple of 2π fills the entire circle eventually is very challenging.

Q5 i) Consider $f_n(x) = x^{2n} = (x^2)^n$

Notice $x^2 \geq 0$ so we get the same limits (if one exists) for $+x$ & $-x$

To put it another way $f_n(x)$ is an <u>even</u> function for all n.

Now if $|x| < 1$ then $|x^2| < 1$ and the larger n is, the smaller $(x^2)^n$ is

$\therefore$ when $|x| < 1$ $f_n(x) \to 0$ as $n \to \infty$

Now when $|x| = 1$, or $x = 1$ or -1 then

$f_n(x) = \left[(\pm 1)^2\right]^n = 1^n = 1$ for all n

$\therefore$ $|x| = 1 \Rightarrow f_n(x) \to 1$ as $n \to \infty$

when $|x| > 1$, $|x^2| > 1$ and as $n \to \infty$

$(x^2)^n$ grows **without** bound.

ii) Notice this is **just our example** from class with a pre-processing step

$x \to \boxed{\text{stretch}} \to 2x$

So when $2x > 1$ get unbounded growth
similarly when $2x < -1$

$\therefore$ $|x| > \frac{1}{2} \Rightarrow f_n(x)$ has no limit

when $2x = 1$ $f_n(x) = 1^n = 1$ $\therefore$ $f_n(x) \to 1$ when $x = \frac{1}{2}$

(Q5) ii) cont'd/

When $2x = -1$ or $x = -\frac{1}{2}$; $f_n(x) = (-1)^n$
and the sequence does not converge
when $|2x| < 1$ or $|x| < \frac{1}{2}$ $|f_n(x)|$
gets smaller and smaller as n increases
and $f_n(x) \to 0$ as $n \to \infty$

(Q6) i) we know $a_n \to L$ and are told
that for $\varepsilon_1 > 0$ we have
$|a_n - L| < \varepsilon_1$ whenever $n > N_1$
Now when we consider $\varepsilon_2 = \frac{\varepsilon_1}{2} > 0$
we find $|a_n - L| < \varepsilon_2$ whenever
$n > N_2$

because $\varepsilon_2 < \varepsilon_1$ we need to go at
least as far as we did for ε_1 and
thus $N_2 \geq N_1$

ii), iii) iv) We know $a_n \to 1$ and $b_n \to 2$
So pick $\varepsilon = m > 0$ we know we can
find N_a so that $n > N_a$ guarantees
$|a_n - 1| < \frac{m}{2}$ and N_b (possibly
quite different from N_a) so that

(Q6) ii), iii) iv) cont'd/

$|b_n - 2| < \frac{m}{2}$ whenever $n > N_b$

Now choose $N = \max\{N_a, N_b\}$

for $n > N$ consider

$$|a_n + b_n - 3| = |a_n - 1 + b_n - 2|$$

$$\leq |a_n - 1| + |b_n - 2|$$

by the triangle inequality.

Since $N \geq N_a$ and N_b we know

$n > N$ gives $|a_n - 1| < \frac{m}{2}$ and $|b_n - 1| < \frac{m}{2}$

$\therefore$ $|a_n + b_n - 3| < \frac{m}{2} + \frac{m}{2} = m = \varepsilon$

This argument works for all $\varepsilon > 0$

$\therefore$ $a_n \to 1$, $b_n \to 2$ implies $a_n + b_n \to 3$

(Q7) i) $\frac{1}{3-x} = \frac{1}{3} \frac{1}{1-\frac{x}{3}} = \frac{1}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right]$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Convergence is given by the GST (pg 87)

when $\left|\frac{x}{3}\right| < 1$ or $|x| < 3$

Q7 ii) $\frac{1}{2-3x} = \frac{1}{2} \frac{1}{1-\frac{3}{2}x} = \frac{1}{2}\left(1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \dots\right)$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{3x}{2}\right)^n$$

The GST theorem ensures convergence when $\left|\frac{3x}{2}\right| < 1$ or $|x| < \frac{2}{3}$

iii) $\frac{1}{3+x} = \frac{1}{3} \frac{1}{1+\frac{x}{3}} = \frac{1}{3} \frac{1}{1-(-\frac{x}{3})} = \frac{1}{3}\left[1 - \frac{x}{3} + \left(\frac{x}{3}\right)^2 - \left(\frac{x}{3}\right)^3 + \dots\right]$

$$= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{3}\right)^n$$