MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 2

1: (a) Let R be the region given by $0 \le y \le 1 - \frac{1}{4}x^2$ and $-2 \le x \le 2$. Find the volume of the solid obtained by revolving R about the x-axis.

Solution: The volume is $V = 2\int_0^2 \pi\left(1 - \frac{1}{4}x^2\right)^2\,dx = 2\pi\int_0^2 1 - \frac{1}{2}x^2 + \frac{1}{16}x^4\,dx = 2\pi\left[x - \frac{1}{6}x^3 + \frac{1}{80}x^5\right]_0^2 = 2\pi\left(2 - \frac{4}{3} + \frac{2}{5}\right) = \frac{32}{15}\pi$.

(b) Let S be the region given by $\frac{1}{4}x^2 - 1 \le y \le 1 - \frac{1}{4}x^2$ and $0 \le x \le 2$. Find the volume of the solid obtained by revolving S about the y-axis.

Solution: Using cylindrical shells, the volume is given by $V = 2\int_0^2 2\pi x\left(1 - \frac{1}{4}x^2\right)\,dx = 4\pi\int_0^2 x - \frac{1}{4}x^2\,dx = 4\pi\left[\frac{1}{2}x^2 - \frac{1}{16}x^4\right]_0^2 = 4\pi(2-1) = 4\pi$.

2: Let R be the (infinitely long) region given by $0 \le y \le \dfrac{1}{1+x^2}$ and $x \ge 0$.

(a) Find the volume of the solid obtained by revolving R about the x-axis.

Solution: The volume is given by $V = \displaystyle\int_{x=0}^{\infty} \pi\,\frac{1}{(1+x^2)^2}\,dx$. We let $\tan\theta = x$ so $\sec\theta = \sqrt{1+x^2}$ and $\sec^2\theta\,d\theta = dx$, and then we obtain $V = \displaystyle\int_{\theta=0}^{\pi/2} \pi\,\frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \pi\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi\int_0^{\pi/2}\frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta = \pi\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi^2}{4}$.

(b) Find the volume of the solid obtained by revolving R about the y-axis.

Solution: Using cylindrical shells, the volume is $V = \displaystyle\int_{x=0}^{\infty} 2\pi x\,\frac{1}{1+x^2}\,dx$. Let $u = 1+x^2$ so that $du = 2x\,dx$, and then $V = \pi\displaystyle\int_{u=1}^{\infty}\frac{1}{u}\,du = \pi\Big[\ln u\Big]_1^{\infty} = \infty$

3: Find the volume of the solid which is obtained by revolving the disc $(x-1)^2 + y^2 \le 1$ about the y-axis.

Solution: Using cylindrical shells, the volume is $V = 2\displaystyle\int_{x=0}^{2} 2\pi x\sqrt{1-(x-1)^2}\,dx$. Let $\sin\theta = x - 1$ so that $\cos\theta = \sqrt{1-(x-1)^2}$ and $\cos\theta\,d\theta = dx$. Then we have $V = 4\pi\displaystyle\int_{\theta=-\pi/2}^{\pi/2}(\sin\theta + 1)\cos\theta\,\cos\theta\,d\theta = 4\pi\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta + \cos^2\theta\,d\theta = 4\pi\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta = 4\pi\left[-\frac{1}{3}\cos^3\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 4\pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = 2\pi^2$.

4: A circular hole of radius 1 is bored through the center of a wooden ball of radius 2. Find the volume of the remaining portion of the ball.

Solution: We provide two solutions. For the first solution, we note that the remaining portion of the ball is in the shape of the solid obtained by revolving the region given by $1 \le y \le \sqrt{4-x^2}$ and $-\sqrt{3} \le x \le \sqrt{3}$ about the x-axis. The cross-section at x is shaped like an annulus (that is a circular disc with a smaller circular hole in the center) with outer radius $\sqrt{4-x^2}$ and inner radius 1. The cross-sectional area is $A(x) = \pi(4-x^2)-\pi = \pi(3-x^2)$. The volume is $V = 2\int_0^{\sqrt{3}} \pi(3-x^2)\,dx = 2\pi\Big[3x - \frac{1}{3}x^3\Big]_0^{\sqrt{3}} = 2\pi\left(3\sqrt{3}-\sqrt{3}\right) = 4\pi\sqrt{3}$.

For the second solution, we note that the remaining portion of the ball is in the shape of the solid obtained by revolving the region given by $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ and $1 \le x \le 2$ about the y-axis, so using cylindrical shells, the volume is $V = 2\int_{x=1}^{2} 2\pi\, x\sqrt{4-x^2}\,dx$. Letting $u = 4-x^2$ so that $du = -2x\,dx$ we obtain $V = \int_{u=3}^{0} -2\pi\sqrt{u}\,du = -2\pi\Big[\frac{2}{3}u^{3/2}\Big]_3^0 = -2\pi\left(-2\sqrt{3}\right) = 4\pi\sqrt{3}$.

5: Find the arclength of the curve $y = e^x$ with $0 \le x \le \ln 2$.

Solution: We have $y' = e^x$ so that arclength is $L = \int_{x=0}^{\ln 2} \sqrt{1+e^{2x}}\,dx$. Let $u = \sqrt{1+e^{2x}}$ so that $u^2 = 1+e^{2x}$ and $2u\,du = 2e^{2x}\,dx$, so $dx = \frac{u}{e^{2x}}\,du = \frac{u}{u^2-1}\,du$. Then we obtain $L = \int_{u=\sqrt{2}}^{\sqrt{5}} \frac{u^2\,du}{u^2-1} = \int_{\sqrt{2}}^{\sqrt{5}} 1+\frac{1}{u^2-1}\,du = \int_{\sqrt{2}}^{\sqrt{5}} 1+\frac{\frac{1}{2}}{u-1}-\frac{\frac{1}{2}}{u+1}\,du = \Big[u+\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\Big]_{\sqrt{2}}^{\sqrt{5}} = \sqrt{5}+\frac{1}{2}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \sqrt{2} - \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$.