MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 4

1: In an electric circuit with a battery, producing a voltage of V volts, a resistor of resistance R ohms and a capacitor of capacitance C farads, the charge $Q(t)$ on the capacitor at time t seconds satisfies the differential equation $R\,Q' + \frac{1}{C}\,Q = V$, and the current in the wire is given by $I(t) = Q'(t)$. Suppose that $V = 24$, $R = 4$ and $C = \frac{1}{12}$, and that $Q(0) = 0$. Find the current $I(t)$ in the circuit at time t.

Solution: We need to solve the DE $4Q' + 12Q = 24$. This is linear, since we can write it as $Q' + 3Q = 6$. An integrating factor is $\lambda = e^{3t}$, and the solution to the DE is $Q(t) = e^{-3t}\displaystyle\int 6\,e^{3t}\,dt = e^{-3t}\left(2\,e^{3t} + c\right) = 2 + c\,e^{-3t}$. To get $Q(0) = 0$ we need $2 + c = 0$ so $c = -2$ and the solution is $Q(t) = 2 - 2\,e^{-3t}$. Thus the current at time t is $I(t) = Q'(t) = 6\,e^{-3t}$.

2: A pot of boiled water, sitting in a room of constant temperature 19°, cools from 100° down to 43° in 30 minutes. When will it cool down to 35°?

Solution: By Newton's Law of Cooling, the temperature $T(t)$ of the water satisfies the DE $T' = k(19 - T)$ for some constant k. This is linear since we can write it as $T' + kT = 19k$. An integrating factor is $\lambda = e^{kt}$ and the solution is $T(t) = e^{-kt}\displaystyle\int 19k\,e^{kt}\,dt = e^{-kt}\left(19\,e^{kt} + c\right) = 19 + c\,e^{-kt}$. Since $T(0) = 100$ we have $19 + c = 100$ so $c = 81$ and so the solution is $T(t) = 19 + 81\,e^{-kt}$. Since $T(30) = 43$ we have $19 + 81e^{-30k} = 43$ so $e^{-30k} = \frac{24}{81} = \frac{8}{27}$, and hence $-30k = \ln\frac{8}{27} = 3\ln\frac{2}{3} = -3\ln\frac{3}{2}$, so $k = \frac{1}{10}\ln\frac{3}{2}$. Finally, we have $T(t) = 35 \iff 19 + 81\,e^{-kt} = 35 \iff 81\,e^{-kt} = 16 \iff e^{-kt} = \frac{16}{81} \iff -kt = \ln\frac{16}{81} = 4\ln\frac{2}{3} = -4\ln\frac{3}{2} \iff t = \dfrac{4\ln\frac{3}{2}}{k} = \dfrac{4\ln\frac{3}{2}}{\frac{1}{10}\ln\frac{3}{2}} = 40$. Thus the water cools to 35° after 40 minutes.

3: The amount $A(t)$ of a radioactive substance decays exponentially with a half-life of 3 seconds. If $A(2) = 20$ then find the time t at which $A(t) = 4$.

Solution: We have $A(t) = A(0)e^{-kt}$ for some constant $k > 0$. Since the half-life is 3 seconds, we have $A(3) = \frac{1}{2}\,A(0)$, that is $A(0)e^{-3k} = \frac{1}{2}\,A(0)$, so $e^{-3k} = \frac{1}{2}$, hence $e^{3k} = 2$, so $3k = \ln 2$, and so $k = \frac{1}{3}\ln 2$. Also, we have $A(2) = 20 \implies A(0)e^{-2k} = 20 \implies A(0) = 20e^{2k} = 20e^{\frac{2}{3}\ln 2} = 20\cdot 2^{2/3}$, and so we have $A(t) = 20\cdot 2^{2/3}e^{-kt}$. Finally, we have $A(t) = 4 \iff 20\cdot 2^{2/3}e^{-kt} = 4 \iff e^{kt} = 5\cdot 2^{2/3} \iff kt = \ln\left(5\cdot 2^{2/3}\right) \iff t = \dfrac{\ln\left(5\cdot 2^{2/3}\right)}{k} = \dfrac{\ln 5 + \frac{2}{3}\ln 2}{\frac{1}{3}\ln 2} = \frac{3\ln 5}{\ln 2} + 2$. Thus $A(t) = 4$ when $t = \frac{3\ln 5}{\ln 2} + 2 \cong 8.97$.

4: A tank initially contains 100 L of pure water. Brine, with a salt concentration 5 gm/L is added at 4 L/min. The solution is kept well mixed and is drained from the tank at a rate of 6 L/min. Find the concentration of salt in the tank when it contains 40 L of solution.

Solution: Since the solution is added at a rate of 4 L/min and drains at a rate of 6 L/min, the volume of solution in the tank decreases at 2 L/min, so the volume is $V(t) = 100 - 2t$. The total amount of salt in the tank $S(t)$ satisfies the DE $S' = 4\cdot 5 - 6\cdot\dfrac{S}{V} = 20 - \dfrac{6S}{100 - 2t} = 20 - \dfrac{3S}{50 - t}$. This DE is linear since we can write it as $S' + \frac{3}{50-t}\,S = 20$. An integrating factor is $\lambda = e^{-3\ln(50-t)} = (50-t)^{-3}$ and the solution is $S(t) = (50-t)^3\displaystyle\int 20\,(50-t)^{-3}\,dt = (50-t)^3\left(10(50-t)^{-2} + c\right) = 10(50-t) + c(50-t)^3$. Since $S(0) = 0$ we have $500 + 125000\,c = 0$ so $c = -\frac{500}{125000} = -\frac{1}{250}$, and so the solution is $S(t) = 10(50-t) - \frac{1}{250}(50-t)^3$. The tank contains 40 L when $100 - 2t = 40$, that is when $t = 30$, and we have $S(30) = 10\cdot 20 - \frac{1}{250}\cdot 20^3 = 200 - 32 = 168$. Thus the concentration of salt is $\frac{168}{40} = 4.2$ gm/L.

5: A tank, in the shape of an inverted cone of radius 1 m and height 4 m, is filled with water. Water then drains from a hole of area 25 cm^2 at the bottom tip of the tank. If the water drains at a velocity of $v = 4\sqrt{y}$ m/s, where y m is the depth of the water in the tank, then find the time at which the tank will be empty.

Solution: Since the water drains at speed $v = 4\sqrt{y}$ from a hole of area $A = \frac{25}{10000} = \frac{1}{400}$ (in m^2), we have $V' = -Av = -\frac{1}{100}\sqrt{y}$, where $V = V(t)$ is the volume of water in the tank at time t. On the other hand, when the water is y m deep, the water in the tank forms a cone of height y and radius $\frac{1}{4}\,y$, so the volume is $V = \frac{1}{3}\,\pi\left(\frac{1}{4}\,y\right)^2 y = \frac{1}{48}\,\pi y^3$, and so we have $V' = \frac{1}{16}\,\pi y^2\,y'$. Equating these two expressions for V' we find that $\frac{1}{16}\,\pi y^2\,y' = -\frac{1}{100}\,y^{1/2}$, so y satisfies the DE $\frac{\pi}{16}\,y^2\,y' = -\frac{1}{100}\sqrt{y}$ which we can write as $y^{\frac{3}{2}}\,dy = -\frac{4}{25\pi}\,dt$. Integrate both sides to get $\frac{2}{5}\,y^{5/2} = -\frac{4}{25\pi}\,t + c$. Put in $y(0) = 4$ to get $c = \frac{2}{5}\cdot 32$, so we have $\frac{2}{5}\,y^{5/2} = \frac{2}{5}\cdot 32 - \frac{4}{25\pi}\,t$, that is $y = \left(32 - \frac{2}{5\pi}\,t\right)^{2/5}$. The tank will be empty when $y = 0$, and this happens when $\frac{2}{5\pi}\,t = 32$, that is when $t = 80\pi$ (so it takes about 4 minutes and 11 seconds for the tank to empty).