Name (print): SOLUTIONS.

Signature: ______________________

I.D. No.: ______________________

MATH 128 = Calculus 2 for the Sciences

Faculty of Mathematics, University of Waterloo
Term Test 2, Fall Term 2006
Date: November 13, 2006
Time: 4:30 pm - 6:20 pm
Instructor: A. Allison

NO ADDITIONAL MATERIAL ALLOWED

Pages: This exam contains 9 pages, including this cover sheet and a page at the end for rough work.

Instructions: Print your name, and write your signature and student ID number at the top of this page. Except for the short answer questions on page 2, correct answers must be fully justified to receive full marks.

Question	Mark
1	/10
2	/10
3	/7
4	/8
5	/10
6	/10
7	/10
Total	/65

1. For the following short answer questions, you do not need to justify your answer.

(a) True or false questions .... Circle the correct answer:

i. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent. TRUE FALSE

ii. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent. TRUE FALSE

iii. If $\sum_{n=1}^{\infty} a_n$ (where $a_n \neq 0$) converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges. TRUE FALSE

iv. $1^x + 2^x + 3^x + \ldots$ is a power series. TRUE FALSE

v. If $\sum_{n=N+1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$ also converges. TRUE FALSE

vi. If $\sum c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$. TRUE FALSE

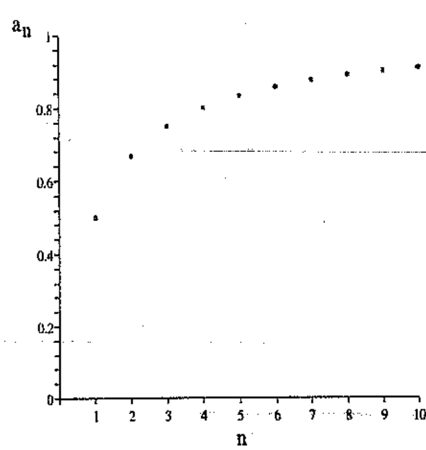
vii. $\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^2}$ is divergent. TRUE FALSE

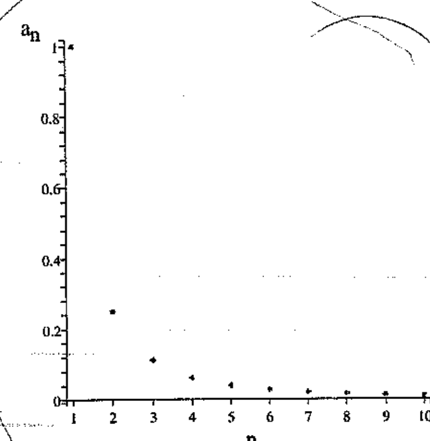
(b) Suppose $\sum a_n = 3$ and s_n is the n^{th} partial sum of the series.

i. What is $\lim_{n\to\infty} a_n$? 0

ii. What is $\lim_{n\to\infty} s_n$? 3

(c) For the sequences $\{a_n\}$ depicted below, which one might correspond to a convergent series, $\sum a_n$? Circle the correct graph.





Typo.

2. (a) $\lim_{n\to\infty} \cancel{a_n =} \frac{\sqrt{n}}{1+\sqrt{n}}$ $= \lim_{n\to\infty} \frac{\sqrt{n}\cdot\frac{1}{\sqrt{n}}}{1+\sqrt{n}\cdot\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1$ 1 1

/2

(b) $\lim_{n\to\infty} \frac{\ln n}{n^3}$ $\leftarrow \frac{\infty}{\infty}$, indeterminate

Let $f(x) = \frac{\ln x}{x^3}$: $\lim_{x\to\infty} \frac{\ln x}{x^3} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{3x^2} = \lim_{x\to\infty} \frac{1}{3x^3} = 0$ 1

/2

$\therefore \lim_{n\to\infty} \frac{\ln n}{n^3} = 0$ 1

(c) Let $a_1 = 4$ and $a_{n+1} = \frac{a_n + 1}{4}$ for $n \geq 1$. Use mathematical induction to show that $\{a_n\}$ converges and find $\lim_{n\to\infty} a_n$.

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$a_1 = 4$, $a_2 = \frac{5}{4}$, $a_3 = \frac{9}{16}$ $\Rightarrow$ a_n seems to be decreasing

1 $\longrightarrow$ Prove $a_{n+1} \leq a_n$ $\forall n \geq 1$:

1 $n=1$: $a_1 = 4$, $a_2 = \frac{5}{4}$, $\therefore a_2 \leq a_1$ ✓

1 $n=k$: Assume $a_{k+1} \leq a_k$.

$n = k+1$: (prove $a_{k+2} \leq a_{k+1}$):

2 Since $a_{k+1} \leq a_k$

$\Rightarrow a_{k+1} + 1 \leq a_k + 1$

$\Rightarrow \underbrace{\frac{a_{k+1}+1}{4}}_{=a_{k+2}} \leq \underbrace{\frac{a_k+1}{4}}_{=a_{k+1}}$ ✓

$\therefore$ $\{a_n\}$ is decreasing and bounded by $1/3$, $\therefore$ $\{a_n\}$ is convergent

1 $\longrightarrow$ Limit $\{a_n\}$: Let $L = \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1}$.

Then $a_{n+1} = \frac{a_n+1}{4}$ $\Rightarrow$ $L = \frac{L+1}{4}$ $\Rightarrow 4L = L+1 \Rightarrow 3L = 1$ $\therefore \boxed{L = \frac{1}{3}}$

3. (a) Write the number $0.\bar{2} = 0.22222\ldots$ as a ratio of integers. (Hint: use geometric series.)

$$0.222\ldots = \underbrace{\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + \cdots}_{1} = \sum_{n=1}^{\infty} \frac{2}{10^n} = \underbrace{\sum_{n=1}^{\infty} 2\left(\frac{1}{10}\right)^n}_{1}$$

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$\longrightarrow$ geometric series (convergent) with $|r| = \frac{1}{10} < 1$, $a = \frac{2}{10} = \frac{1}{5}$

$$\Rightarrow \sum_{n=1}^{\infty} 2\left(\frac{1}{10}\right)^n = \frac{1/5}{1-\frac{1}{10}} = \frac{1/5}{9/10} = \frac{10}{5(9)} = \boxed{\frac{2}{9}} \quad 1$$

1

check: $9\overline{)2.00}$ = .22... ; 18, 20, 18, 2 ✓

(b) For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

3

Option 1: $\sum_{n=1}^{\infty} (\ln x)^n \leftarrow$ geometric series, convergent if

1 — $|r| = |\ln x| < 1$

$\Rightarrow -1 < \ln x < 1$ — 1

$\therefore \boxed{e^{-1} < x < e}$ — 1

option 2: apply Ratio Test: let $a_n = (\ln x)^n$, $a_{n+1} = (\ln x)^{n+1}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \underbrace{\left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| = |\ln(x)|}_{1} < 1 \Rightarrow -1 < \ln x < 1$$

$\Rightarrow e^{-1} < x < e$ — 1

must check endpoints: (if geom. series ~~property not~~ recognized)

1 { $\frac{1}{2}$ $x = e^{-1}$: $\sum_{n=1}^{\infty} \left(\ln \frac{1}{e}\right)^n = \sum (\underbrace{\ln 1}_{0} - \underbrace{\ln e}_{1})^n = \sum_{n=1}^{\infty} (-1)^n \rightarrow$ divergent

$\frac{1}{2}$ $x = e$: $\sum_{n=1}^{\infty} (\underbrace{\ln e}_{1})^n = \sum_{n=1}^{\infty} 1^n \rightarrow$ divergent $\quad \therefore \boxed{e^{-1} < x < e}$

4. (a) Use partial fractions to find the exact value of $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$.

/4

$$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1} = \frac{An - A + Bn}{n(n-1)} \Rightarrow 1 = n(A+B) - A \Rightarrow \boxed{A = -1}$$

$$\Rightarrow \boxed{B = 1}$$

(2) Let $S_N = \displaystyle\sum_{n=3}^{N} -\frac{1}{n} + \frac{1}{n-1}$

(1) $= \left(-\cancel{\frac{1}{3}} + \frac{1}{2}\right) + \left(-\cancel{\frac{1}{4}} + \cancel{\frac{1}{3}}\right) + \left(-\cancel{\frac{1}{5}} + \cancel{\frac{1}{4}}\right) + \dots + \left(-\cancel{\frac{1}{N-1}} + \cancel{\frac{1}{N-2}}\right) + \left(-\frac{1}{N} + \cancel{\frac{1}{N-1}}\right)$

$= \frac{1}{2} - \frac{1}{N}$

($\frac{1}{2}$) $\rightarrow \frac{1}{2}$ as $N \rightarrow \infty$

$$\therefore \boxed{\sum_{n=3}^{\infty} \frac{1}{n(n-1)} = \frac{1}{2}}$$ ($\frac{1}{2}$)

(b) Use the Comparison Test to determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}}$.

/4

(1) Since $|\cos n| \le 1$, $n \ge 1$,

$\Rightarrow \dfrac{|\cos n|}{n^{3/2}} \le \dfrac{1}{n^{3/2}}$ $(n^{3/2} > 0$, actually, $n^{3/2} \ge 1)$

(1) Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series $(p = 3/2 > 1)$,

(1) then $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^{3/2}}$ converges by the CT,

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}}$ converges absolutely

and thus $\boxed{\text{converges.}}$ (1)

5. (a) Use the Limit Comparison Test to determine the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$.

(1)

Let $a_n = \frac{n^2}{n^3+4} \approx \frac{n^2}{n^3} = \frac{1}{n}$ for large n. $\Rightarrow$ let $b_n = \frac{1}{n}$.

$$\Rightarrow \frac{a_n}{b_n} = \frac{\frac{n^2}{n^3+4}}{\frac{1}{n}} = \frac{n^3}{n^3+4} = \frac{1}{1+4/n^3} \to 1 \in (0,\infty), \text{ as } n \to \infty$$

(1) (1) (1)

6

$\therefore$ by LCT, since $\sum b_n$ is a divergent p-series ($p=1$),

(1)

then $\boxed{\sum a_n \text{ also diverges.}}$

(1)

(b) Use the Alternating Series Test to determine whether $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$ converges absolutely, converges conditionally, or diverges.

4

Let $a_n = (-1)^{n-1} n^{-1/3} \Rightarrow |a_n| = n^{-1/3} \to 0$ as $n \to \infty$ (1)

(1) $\Rightarrow \sum a_n$ converges by AST.

or: since $n+1 > n$, $n \geq 1$

$\Rightarrow (n+1)^{1/3} > n^{1/3}$

$\Rightarrow \frac{1}{(n+1)^{1/3}} < \frac{1}{n^{1/3}}$.

$\therefore$ a_n decreasing,

and $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$

(1) Since $\sum |a_n| = \sum n^{-1/3} = \sum \frac{1}{n^{1/3}}$ is a divergent p-series ($p = \frac{1}{3} < 1$),

$\Rightarrow \boxed{\sum a_n \text{ is conditionally convergent}}$

(1)

6. Use the Integral Test to determine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

/10 Let $a_n = \frac{1}{n(\ln n)^2} = f(n)$. Then $f(x) = \frac{1}{x(\ln x)^2}$

- Since $x \geq 2 \Rightarrow x(\ln x)^2 > 0 \Rightarrow \boxed{f(x) > 0}$ and $f(x)$ is also $\boxed{\text{continuous}}$. 1 1

- Since x increasing and so is $\ln x$ increasing, then $(\ln x)^2$ increases, thus $x(\ln x)^2$ increases. Hence $f = \frac{1}{x(\ln x)^2}$ $\boxed{\text{decreases.}}$ 1

OR: $f' = \frac{0 \cdot x(\ln x)^2 - 1\left((\ln x)^2 + x \cdot 2(\ln x)\frac{1}{x}\right)}{x^2(\ln x)^4} = \frac{-(\ln x + 2)}{x^2 (\ln x)^4} < 0$

$\Rightarrow$ $\boxed{f \text{ is decreasing}}$, (since terms in ( ) are all positive since $x \geq 2$).

$\Rightarrow$ IT can be applied:

$\int_2^{\infty} \frac{dx}{x(\ln x)^2}$: Let $\underbrace{u = \ln x \Rightarrow du = \frac{1}{x}dx}_{1}$; $\begin{cases} x = 2 \Rightarrow u = \ln 2, \\ x \to \infty \Rightarrow u \to \infty \end{cases}$

$\Rightarrow \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \boxed{\lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u^2}} = \lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{b}$ 3 1

$= \lim_{b\to\infty}\left[-\frac{1}{b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$ 1

Since $\int_2^{\infty} f(x)\,dx$ converges,

So does $\boxed{\sum a_n \text{ converge}}$ by the IT.

1

* -1 if limit not used *

$-\frac{1}{2}$ if "$\sum a_n \to \frac{1}{\ln 2}$"

7. Find the radius of convergence and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}$.

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Let $a_n = \frac{(x+1)^n}{n2^n} \Rightarrow a_{n+1} = \frac{(x+1)^{n+1}}{(n+1)2^{n+1}}$. Use Ratio Test:

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \overbrace{\left|\frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(x+1)^n}\right|}^{1} = \frac{|x+1|}{2} \underbrace{\frac{n}{n+1}}_{1} = \frac{|x+1|}{2}\left(\frac{1}{1+\frac{1}{n}}\right)$$

$$\left.\longrightarrow \frac{|x+1|}{2} \text{ as } n \to \infty \right\} 1$$

If $\frac{|x+1|}{2} < 1$, $\sum a_n$ converges, by the RT.

$\Rightarrow |x+1| < 2 \Rightarrow \boxed{R = 2}$ — 1

$\Rightarrow -2 < x+1 < 2$

$\Rightarrow -3 < x < 1$ — 1

endpoints:

1 - $x = -3 \Rightarrow \sum a_n = \sum \frac{(-2)^n}{n2^n} = \sum \frac{(-1)^n}{n} \rightarrow$ conditionally convergent (AST; p-series: $p=1$) — 1

(since $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$, divergent p-series, $p=1$; and $\frac{1}{n} \to 0$ as $n \to \infty$, $\sum \frac{(-1)^n}{n}$ conv. conditionally by AST)

1 - $x = 1$: $\Rightarrow \sum a_n = \sum \frac{1}{n}$, divergent p-series ($p=1$) — 1

$\therefore$ IOC: $\boxed{-3 \le x < 1}$ or $x \in \boxed{[-3, 1)}$

OR

1