MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 8

1: Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(1-4x)^n}{n\,2^n}$.

Solution: Let $a_n = \dfrac{(1-4x)^n}{n\,2^n}$. Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|1-4x|^{n+1}}{(n+1)\,2^{n+1}}\,\frac{n\,2^n}{|1-4x|^n} = \lim_{n\to\infty} \tfrac{1}{2}|1-4x|\left(\tfrac{n}{n+1}\right) = \tfrac{1}{2}\,\big|1-4x\big| = 2\,\big|x-\tfrac{1}{4}\big|$. By the Ratio Test, $\sum a_n$ converges when $\big|x-\frac{1}{4}\big| < \frac{1}{2}$ and diverges when $\big|x-\frac{1}{4} > \frac{1}{2}$. Also, we have $\big|x-\frac{1}{4}\big| = \frac{1}{2}$ when $x = -\frac{1}{4}$ or $\frac{3}{4}$. When $x = -\frac{1}{4}$, $a_n = \frac{2^n}{n\,2^n} = \frac{1}{n}$ so $\sum a_n$ diverges, and when $x = \frac{3}{4}$, $a_n = \frac{(-2)^n}{n\,2^n} = \frac{(-1)^n}{n}$ so $\sum a_n$ converges by the A.S.T. Thus the interval of convergence is $\big(-\frac{1}{4}, \frac{3}{4}\big]$.

2: Find the Taylor series centered at 0 for $f(x) = \dfrac{1}{x^2+3x+2}$.

Solution: $f(x) = \dfrac{1}{x^2+3x+2} = \dfrac{1}{x+1} - \dfrac{1}{x+2} = \dfrac{1}{1+x} - \dfrac{\frac{1}{2}}{1+\frac{x}{2}} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n - \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(1-\tfrac{1}{2^{n+1}}\right)x^n$, or in alternate notation $f(x) = \frac{1}{2} - \frac{3}{4}\,x + \frac{7}{8}\,x^2 - \frac{15}{16}\,x^3 + \frac{31}{32}\,x^4 - \cdots$. .

3: Find the Taylor Polynomial of degree 5 centered at 0 for $f(x) = \dfrac{e^x}{1+x}$.

Solution:
$$\begin{aligned}
f(x) &= e^x\,\frac{1}{1+x}\\
&= \left(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\tfrac{1}{24}x^4+\tfrac{1}{120}x^5+\cdots\right)\left(1-x+x^2-x^3+x^4-x^5-\cdots\right)\\
&= 1+\left(-1+1\right)x+\left(1-1+\tfrac{1}{2}\right)x^2+\left(-1+1-\tfrac{1}{2}+\tfrac{1}{6}\right)x^3+\left(1-1+\tfrac{1}{2}-\tfrac{1}{6}+\tfrac{1}{24}\right)x^4\\
&\qquad+\left(-1+1-\tfrac{1}{2}+\tfrac{1}{6}-\tfrac{1}{24}+\tfrac{1}{120}\right)x^5+\cdots\\
&= 1+0x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+\tfrac{9}{24}x^4-\tfrac{44}{120}x^5+\cdots
\end{aligned}$$
so the Taylor polynomial of degree 5 is $P_5(x) = 1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{3}{8}x^4-\frac{11}{30}x^5$.

4: Find the Taylor Polynomial of degree 5 centered at 0 for $f(x) = (1+2x)^{3/2}\sin x$.

Solution: $f(x) = (1+2x)^{3/2}\sin x$
$$\begin{aligned}
&= \left(1+\tfrac{3}{2}(2x)+\tfrac{(\frac{3}{2})(\frac{1}{2})}{2!}(2x)^2+\tfrac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3!}(2x)^3+\tfrac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{4!}(2x)^4+\cdots\right)\left(x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5+\cdots\right)\\
&= \left(1+3x+\tfrac{3}{2}x^2-\tfrac{1}{2}x^3+\tfrac{3}{8}x^4+\cdots\right)\left(x+0x^2-\tfrac{1}{6}x^3+0x^4+\tfrac{1}{120}x^5+\cdots\right)\\
&= x+(0+3)x^2+\left(-\tfrac{1}{6}+0+\tfrac{3}{2}\right)x^3+\left(0-\tfrac{1}{2}+0-\tfrac{1}{2}\right)x^4+\left(\tfrac{1}{120}+0-\tfrac{1}{4}+0+\tfrac{3}{8}\right)x^5+\cdots\\
&= x+3x^2+\tfrac{8}{6}x^3-x^4+\tfrac{16}{120}x^5+\cdots
\end{aligned}$$
so the Taylor polynomial of degree 5 is $P_5(x) = x+3x^2+\frac{4}{3}x^3-x^4+\frac{2}{15}x^5$.

5: Find the Taylor Polynomial of degree 4 centered at 0 for $f(x) = \dfrac{\ln(1+x)}{\tan^{-1} x}$.

Solution: We have $\dfrac{\ln(1+x)}{\tan^{-1} x} = \dfrac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots}{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots} = \dfrac{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \cdots}{1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \cdots}$. We use long division:

$$
\begin{array}{r l}
 & 1 - \tfrac{1}{2}x + \tfrac{2}{3}x^2 - \tfrac{5}{12}x^3 + \tfrac{2}{9}x^4 + \cdots \\ \cline{2-2}
1 - \tfrac{1}{3}x^2 + \tfrac{1}{5}x^4 - \cdots \Big) & 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \tfrac{1}{5}x^5 - \cdots \\
 & 1 + 0\,x - \tfrac{1}{3}x^2 + 0\,x^3 + \tfrac{1}{5}x^4 + \cdots \\ \cline{2-2}
 & -\tfrac{1}{2}x + \tfrac{2}{3}x^2 - \tfrac{1}{4}x^3 + 0\,x^4 + \cdots \\
 & -\tfrac{1}{2}x + 0\,x^2 + \tfrac{1}{6}x^3 + 0\,x^4 + \cdots \\ \cline{2-2}
 & \tfrac{2}{3}x^2 - \tfrac{5}{12}x^3 + 0\,x^4 + \cdots \\
 & \tfrac{2}{3}x^2 + 0\,x^3 - \tfrac{2}{9}x^4 + \cdots \\ \cline{2-2}
 & -\tfrac{5}{12}x^3 + \tfrac{2}{9}x^4 + \cdots
\end{array}
$$

Thus the Taylor polynomial of degree 4 is $T_4(x) = 1 - \frac{1}{2}\,x + \frac{2}{3}\,x^2 - \frac{5}{12}\,x^3 + \frac{2}{9}\,x^4$.