MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 4 SOLUTIONS

Due **Friday, October 13** in Drop Box 9 before class
To receive full marks, correct answers must be fully justified.

1. Show that $y = \frac{2}{3}\sin x + \frac{4}{5}\cos x + \frac{1}{5}e^{2x}$ is a solution of the differential equation (DE) $y'' + y = e^{2x}$.

   Solution: From the given expression for y:

$$\begin{aligned} \Rightarrow y' &= \frac{2}{3}\cos x - \frac{4}{5}\sin x + \frac{2}{5}e^{2x} \Rightarrow y'' = -\frac{2}{3}\sin x - \frac{4}{5}\cos x + \frac{4}{5}e^{2x} \\ \Rightarrow y'' + y &= -\frac{2}{3}\sin x - \frac{4}{5}\cos x + \frac{4}{5}e^{2x} + \frac{2}{3}\sin x + \frac{4}{5}\cos x + \frac{1}{5}e^{2x} \end{aligned}$$

   Canceling terms on the right-hand side and then simplifying, we have:

$$y'' + y = \left(\frac{4}{5} + \frac{1}{5}\right)e^{2x} = e^{2x},$$

   as desired.

2. Find all solutions of the form $y = Ax + B + \frac{C}{x}$ to the DE $x(y')^2 + y = \frac{3}{4}x + 1 + \frac{4}{x^3}$.

   Solution: From the given expression for y:

$$\begin{aligned} \Rightarrow y' &= A - \frac{C}{x^2} \Rightarrow (y')^2 = \left(A - \frac{C}{x^2}\right)^2 = A^2 - \frac{2AC}{x^2} + \frac{C^2}{x^4} \\ \Rightarrow x(y')^2 + y &= x\left(A^2 - \frac{2AC}{x^2} + \frac{C^2}{x^4}\right) + Ax + B + \frac{C}{x} = \frac{3}{4}x + 1 + \frac{4}{x^3} \\ \Rightarrow \quad & xA^2 - \frac{2AC}{x} + \frac{C^2}{x^3} + Ax + B + \frac{C}{x} = \frac{3}{4}x + 1 + \frac{4}{x^3} \end{aligned}$$

   Collecting the left-hand side in coefficients of powers of x:

$$\Rightarrow \quad x(A^2 + A) + \frac{1}{x}(-2AC + C) + \frac{1}{x^3}(C^2) + B = \frac{3}{4}x + 1 + \frac{4}{x^3}$$

   Matching coefficients of powers of x on both sides:

   x^0: $B = 1$,

   $\frac{1}{x^3}$: $C^2 = 4$ so that $C = \pm 2$,

   $\frac{1}{x}$: $-2AC + C = C(-2A + 1) = 0$ so that $A = \frac{1}{2}$, since we already know $C \neq 0$,

   x^1: $A^2 + A = \frac{3}{4}$: our value for $A = \frac{1}{2}$ also satisfies this condition

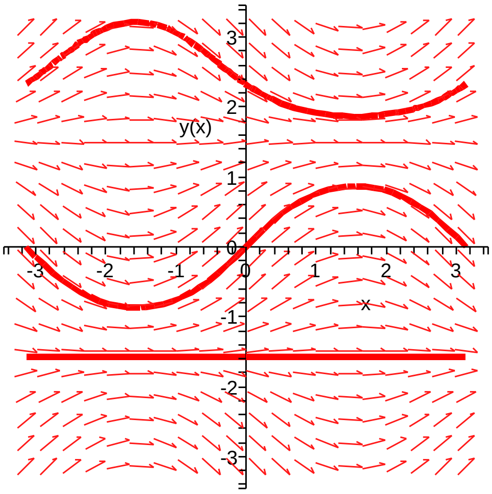
   Therefore, we see there are two solutions when we back-substitute the values for A, B, and C into the given expression for y:

$$\begin{aligned} y &= \frac{x}{2} + 1 + \frac{2}{x}, \text{ and} \\ y &= \frac{x}{2} + 1 - \frac{2}{x} \end{aligned}$$

3. Below is the slope (or direction) field of $y' = \cos x \cos y$.

   (a) On the given slope field, sketch the graphs of the solutions that satisfy the following initial conditions: $y(0) = 0, y(2) = -\pi/2$, and $y(2) = 2$. Clearly label the three curves.

   Solution:
   $y(2) = 2$: top curve
   $y(0) = 0$: middle curve
   $y(2) = -\pi/2$: bottom curve



   (b) Find all of the equilibrium solutions of the DE.

   Solution: Equilibrium solutions are constant-valued solutions $y = y^*$ of the DE $y' = \cos x \cos y^* = 0$. Here, this can happen only when $\cos y^* = 0$, which occurs for $y^* = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$ (*i.e.*, odd integer multiples of $\pi/2$).

4. Find the general solution to each of the following DEs.

   (a) $\dfrac{dy}{dx} + e^{x+y} = 0$.

   Solution:

$$\begin{aligned} \frac{dy}{dx} + e^{x+y} = 0 \Rightarrow \frac{dy}{dx} &= -e^{x+y} = -e^x e^y \Rightarrow e^{-y} dy = -e^x dx \Rightarrow \int e^{-y} dy = \int -e^x dx \\ \Rightarrow -e^{-y} &= -e^x + C \Rightarrow e^{-y} = e^x - C \Rightarrow -y = \ln|e^x - C| \\ \therefore y &= -\ln|e^x - C| \end{aligned}$$

   Note: the final answer can have $+C$ instead, as long as you've renamed it elsewhere.

   (b) $\dfrac{dx}{d\theta} = \dfrac{e^x \cos^2\theta}{x \csc\theta}$.

   Solution:

$$\begin{aligned} \frac{x}{e^x} dx &= \frac{\cos^2\theta}{\csc\theta} d\theta \Rightarrow x e^{-x} dx = \frac{\cos^2\theta}{\frac{1}{\sin\theta}} d\theta = \sin\theta \cos^2\theta d\theta \\ \Rightarrow \int x e^{-x} dx &= \int \sin\theta \cos^2\theta d\theta \end{aligned}$$

   where we use integration by parts for the integral on the left:

$$u = x \Rightarrow du = dx; dv = e^{-x} \Rightarrow v = -e^{-x},$$

and we let $w = \cos\theta \Rightarrow dw = -\sin\theta d\theta$ on the right:

$$\begin{aligned}
\Rightarrow -xe^{-x} + \int e^{-x}dx &= -\int w^2 dw = -\left(\frac{w^3}{3} + C_1\right) \\
\Rightarrow -xe^{-x} - e^{-x} &= -\frac{\cos^3\theta}{3} + C_2 && (C_2 = -C_1) \\
\therefore xe^{-x} + e^{-x} &= e^{-x}(x+1) = \frac{\cos^3\theta}{3} + C && (C = -C_2)
\end{aligned}$$

(where factoring the left-hand side here is not necessary).

(c) $(1+x^2)y' + 2xy = \frac{1}{1+x^2}$.

Solution: Rearrange the DE in standard linear form: $y' + P(x)y = Q(x)$:

$$\begin{aligned}
y' + \frac{2x}{1+x^2}y &= \frac{1}{(1+x^2)^2} \Rightarrow P(x) = \frac{2x}{1+x^2} \\
\Rightarrow \text{the integrating factor } I(x) &= e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2 \\
\Rightarrow (1+x^2)y' + (1+x^2)\frac{2x}{1+x^2}y &= (1+x^2)\frac{1}{(1+x^2)^2} \\
\Rightarrow \frac{d}{dx}((1+x^2)y) &= \frac{1}{1+x^2} \\
\Rightarrow \int \frac{d}{dx}((1+x^2)y)dx &= \int \frac{1}{1+x^2}dx \\
\Rightarrow (1+x^2)y &= \tan^{-1}x + C \\
\therefore y &= \frac{\tan^{-1}x + C}{1+x^2}
\end{aligned}$$

5. Solve the following initial value problems.

(a) $\frac{dL}{dt} = kL^2\ln t,\ L(1) = -1$.

Solution: Rearrange the DE by separating variables and then integrate:

$$\frac{dL}{L^2} = k\ln t dt \Rightarrow \int \frac{dL}{L^2} = \int k\ln t dt$$

where we use integration by parts for the integral on the right:

$$u = \ln t \Rightarrow du = dt/t; dv = dt \Rightarrow v = t,$$

$$\begin{aligned}
\Rightarrow -\frac{1}{L} &= k\left(t\ln t - \int dt\right) = kt\ln t - kt + C \\
\Rightarrow \frac{1}{L} &= -kt\ln t + kt - C \\
\Rightarrow L &= \frac{1}{-kt\ln t + kt - C} \\
L(1) = -1 \Rightarrow -1 &= \frac{1}{-k(1)\ln 1 + k(1) - C} = \frac{1}{k - C} \Rightarrow k - C = -1 \Rightarrow C = k+1 && (\ln 1 = 0) \\
\therefore L &= \frac{1}{-kt\ln t + kt - k - 1}
\end{aligned}$$

(b) $xy' - \frac{y}{x+1} = x,\ y(1) = 0,\ x > 0.$

Solution: Rearranging the DE into linear form:

$$y' - \frac{y}{x(x+1)} = 1 \Rightarrow P(x) = \frac{-1}{x(x+1)} \Rightarrow I(x) = e^{-\int \frac{dx}{x(x+1)}} = e^{-\int \frac{A}{x} + \frac{B}{x+1} dx}$$

using partial fractions, we now solve for A and B to determine $I(x)$.

$$\begin{aligned}
\frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)}{x(x+1)} + \frac{Bx}{x(x+1)} \\
\Rightarrow 1 = Ax + A + Bx &= x(A+B) + A \\
\Rightarrow \{1 = A, 0 = A + B\} & \quad \Rightarrow B = -A = -1 \\
\Rightarrow I(x) = e^{-\int \frac{1}{x} - \frac{1}{x+1} dx} &= e^{-(\ln x - \ln(x+1))} = e^{-\ln x} e^{\ln(x+1)} = e^{\ln x^{-1}}(x+1) = \frac{x+1}{x} \\
\Rightarrow \frac{x+1}{x} y' - \frac{x+1}{x} \frac{y}{x(x+1)} &= \frac{x+1}{x} \\
\Rightarrow \frac{x+1}{x} y' - \frac{y}{x^2} &= 1 + \frac{1}{x} \\
\Rightarrow \frac{d}{dx}\left[\left(\frac{x+1}{x}\right) y\right] &= 1 + \frac{1}{x} \\
\Rightarrow \int \frac{d}{dx}\left[\left(\frac{x+1}{x}\right) y\right] dx &= \int 1 + \frac{1}{x} dx \\
\Rightarrow \left(\frac{x+1}{x}\right) y &= x + \ln x + C \qquad \text{(can apply IC here, or later)} \\
\Rightarrow y &= \frac{x + \ln x + C}{\frac{x+1}{x}} = \frac{x^2 + x\ln x + Cx}{x+1} \\
y(1) = 0 \Rightarrow 0 = \frac{1 + \ln 1 + C}{2} &= \frac{1 + C}{2} \Rightarrow C + 1 = 0 \Rightarrow C = -1 \qquad (\ln 1 = 0) \\
\therefore y &= \frac{x^2 + x\ln x - x}{x+1}
\end{aligned}$$