Midterm Fall/05 Solutions

Notes on Marking: Marks are awarded for precise statements. Part marks are **not** awarded for vague statements.

Contents

Total here is 12 pages.

1 Homogeneous System

4 marks total

Describe all solutions of $Ax = 0$ in parametric form, where

$$A = \begin{pmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{pmatrix}.$$

Solution:

After the row operation of adding twice row 2 to row 1, the reduced row echelon form of A is ***2 marks***

$$\begin{pmatrix} 1 & 0 & -5 & -7 \\ 0 & 1 & 2 & -6 \end{pmatrix}.$$

Hence the solution in parametric form is: ***2 marks***

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = u \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 7 \\ 6 \\ 0 \\ 1 \end{pmatrix},$$

where u and v are free and represent the parameters. (If not in parametric form, then one mark is deducted.)

2 Consistency

6 marks total

Suppose a system of linear equations has a 8×6 *augmented* matrix whose 6-th column is a pivot column. Is the system consistent? Why or why not? Explain carefully.

Solution:

Detailed explanation is needed.

No. Because having the 6-th column (which is the rightmost column) as a pivot column implies that the augmented matrix has a row:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

where $b \neq 0$, thus this linear system is inconsistent.

3 Span

7 marks total

1. Define the span of a set of vectors.
2. Let
$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 6 \\ 12 \\ h \end{pmatrix}.$$
For what values of h is v_3 in Span$\{v_1, v_2\}$.
3. Let
$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ 9 \\ h \end{pmatrix}.$$
For what values of h is v_3 in Span$\{v_1, v_2\}$.
4. Let
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}. v_4 = \begin{pmatrix} 4 \\ 8 \\ h \end{pmatrix}.$$
For what values of h is v_4 in Span$\{v_1, v_2, v_3\}$.

Solution:

Note, that only a value for h without justification is not worth any marks. Also, having a nontrivial solution to the corresponding homogeneous system for number 2, does not mean that v_3 is in the requested span.

1. ***1 marks***

   The span of a set of vectors is a set which consists of all possible linear combinations of these vectors:

   $$\text{span}\{v_1, v_2, \dots, v_n\} = \{c_1 v_1 + c_2 v_2 + \dots + c_n v_n | c_i \text{ scalar}, \ i = 1, 2, \dots, n\}$$

2. ***2 marks***

Since v_1 and v_2 are collinear (i.e. $v_2 = \alpha v_1$, $\alpha = 2$), then $\text{span}\{v_1, v_2\}$ is an entire line in the direction of either of these vectors. For v_3 to lie in the span, it has to be of the form $v_3 = \beta v_1$. Equating the first two components implies $\beta = 6$. Hence $h = 18$.

3. ***2 marks***

For v_3 to lie in the $\text{span}\{v_1, v_2\}$, there must exist scalars c_1 and c_2 such that $v_3 = c_1 v_1 + c_2 v_2$. To find whether such scalars exist consider the linear system

$$Ac = v_3,$$

where $A = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and $c = \begin{pmatrix} c_1 & c_2 \end{pmatrix}^T$. Doing the first step of row reduction on the augmented matrix yields the following:

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 3 & 9 & h \end{pmatrix}$$

The second row immediately indicates that the system is inconsistent independent on what the value of h is. Hence, there are **NO** values h can take such that v_3 is in the $\text{span}\{v_1, v_2\}$.

4. ***2 marks***

One can observe that vectors v_1, v_2, v_3 are linearly independent, e.g. because there will be a pivot in each column and each row. Hence they span $\Re^3$. This implies that any vector in $\Re^3$ is in the span, thus h can take any value.

4 Linear Independence

7 marks total

1. Define linear independence of a set of vectors in $\Re^n$.

2. Suppose $u, v \in \Re^n$ are linearly independent and $a, b, c, d \in \Re$. Prove that $\{au + bv, cu + dv\}$ is a linearly independent set if and only if $ad \neq bc$.

Solution:

1. ***2 marks***

Vectors $v_1, v_2, \ldots, v_m \in \Re^n$ are linearly indepedent if the equation

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = 0$$

has only the trivial solution, i.e. $c_i = 0, i = 1, 2, \ldots, m$. (Equivalently, no vector v_i is a linear combination of the remaining other vectors. Equivalently, no vector v_i is in the span of the remaining other vectors.)

2. ***5 marks***

To prove the 'only if' direction, we can show that:
if $ad = bc$, then vectors in the set are linearly dependent.
Assume $ad = bc$, and assume also, that both $b \neq 0$ and $d \neq 0$. Consider $d(au+bv)+(-b)(cu+dv) = (ad-bc)u+(db-bd)v = 0$, hence the vectors $au + bv$ and $cu + dv$ are linearly dependent, as there exist nonzero scalars such that the linear combination of the vectors is 0.
For the case when $b = d = 0$, we have $au + bv = au$ and $cu + dv = cu$ which are collinear and hence linearly dependent.

To prove the 'if' direction, we consider

$$g(au + bv) + h(cu + dv) = 0$$

for $g, h \in \Re$. We can rewrite this as

$$(ga + hc)u + (gb + hd)v = 0.$$

By using the fact, that u and v are linearly indepedent, we have $ga+hc = 0$ and $gb + hd = 0$, which yield:

$$\begin{aligned} 0 &= (ga + hc)d = (ga)d + (hd)c = (ga)d - (gb)c = g(ad - bc), \\ 0 &= (gb + hd)a = (ga)b + (hd)a = -(hc)b + (hd)a = h(ad - bc). \end{aligned}$$

Then $ad \neq bc$ implies $g = h = 0$, thus $au + bv$ and $cu + dv$ are linearly indepedent. ■

Alternate Proof
Let $x = au + bv, \ y = cu + dv$.
Suppose that

$$x := au + bv, \quad y := cu + dv \tag{1}$$

are linear dependent, i.e. $\alpha x+\beta y = 0$, for some scalars α, β not both zero. We need to show this is equivalent to

$$ac = bd. \tag{2}$$

Without loss of generality, we can assume that $\alpha \neq 0$. (We can proceed similarly if $\beta \neq 0$.) And, we can assume that $x \neq 0$, by the linear independence of u, v. (Otherwise, this is equivalent to $a = b = 0$ and $ad = bc$.)
Therefore, (1) holds
if and only if

$$x + \gamma y = 0, \qquad \gamma = \beta/\alpha \neq 0$$

if and only if

$$(a + \gamma c)u + (b + \gamma d)v = 0, \quad \text{for some } \gamma \neq 0$$

if and only if

$$(a+\gamma c) = -(b+\gamma d) = 0, \quad \text{for some } \gamma \neq 0$$

if and only if

$$a = -\gamma c, \quad b = -\gamma d, \quad \text{for some } \gamma \neq 0$$

if and only if

$$-\gamma ad = -\gamma bc, \quad \text{for some } \gamma \neq 0$$

if and only if

$$ad = bc.$$

■

5 Linear Combinations

7 marks total

Suppose that $S = \{v_1, \dots v_k, v\}$ is a set of vectors in $\Re^n$ and that v is a linear combination of $v_1, \dots, v_k$. If $S' = \{v_1, \dots v_k\}$, prove that $\text{span}\,(S) = \text{span}\,(S')$.

Solution:

To prove that the two sets are equal we have to prove that every vector in the first set lies in the second and vice versa.

We start by proving that $x \in \text{span}\,(S)$ implies $x \in \text{span}\,(S'), \forall x$.
By definition of the span, $x \in \text{span}\,(S)$ is equivalent to $x = c_1 v_1 + \dots + c_k v_k + cv$, for some scalars $c_1, \dots, c_k, c$. But since v is a linear combination of $v_1, \dots, v_k$, then $v = b_1 v_1 + \dots + b_k v_k$ for some other scalars $b_1, \dots, b_k$. Hence,

$$\begin{aligned} x &= c_1 v_1 + \dots + c_k v_k + c(b_1 v_1 + \dots + b_k v_k) \\ &= (c_1 + cb_1)v_1 + \dots + (c_k + cb_k)v_k, \end{aligned}$$

which means that x is a linear combination of vectors $v_1, \dots, v_k$ and thus $x \in \text{span}\,(S')$.

Conversely, if $x \in \text{span}\,(S')$ then $x = b_1 v_1 + \dots + b_k v_k$ for some scalars $b_1, \dots, b_k$, or

$$x = b_1 v_1 + \dots + b_k v_k = b_1 v_1 + \dots + b_k v_k + 0v.$$

Hence x can be represented as a linear combination of vectors $v_1, \dots, v_k, v$ and thus $x \in \text{span}\,(S)$. ■

6 Linear Transformation

7 marks total

Let $T : \Re^2 \to \Re^2$ be a linear transformation that maps the vector u to the vector x and the vector v to the vector y, i.e.

$$u = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \to x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \to y = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Find the images under T of:

1. $3u$
2. $2v$
3. $3u + 2v$

Solution:

We use the linear properties of the transformation T to establish the following results:

1. ***2 marks***

$$T(3u) = 3T(u) = 3x = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

2. ***2 marks***

$$T(2v) = 2T(v) = 2y = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

3. ***3 marks***

$$T(3u + 2v) = T(3u) + T(2v) = 3x + 2y = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

7 Complex Linear System

12 marks total

Define the matrix A and vector b with complex coefficients as:

$$A = \begin{pmatrix} 1 & i & 1-2i \\ 2 & -1+2i & 1-5i \\ -1 & 0 & -2+3i \end{pmatrix}, \qquad b = \begin{pmatrix} -i \\ 2-2i \\ -i \end{pmatrix}.$$

Consider the equation

$$Ax = b. \tag{3}$$

1. (4 marks) Write down the coefficient matrix and the agumented matrix for the linear system in (3). Then, write down the augmented matrix with the additional check column.

2. (4 marks) Suppose that you made a mistake in calculating the check column and you used $\begin{pmatrix} 4 \\ -i \\ 5 \end{pmatrix}$ instead. What would you get in the new check column position after completing the first pivot in column one, i.e. after obtaining zeros below the $(1,1)$ element in column 1?

3. (4 marks) Determine the *general solution* to the linear system $Ax = b$ using row reductions, i.e. by finding the RREF. You should use the (correct of course) check column at each step of your calculations.

4. (4 marks) **BONUS:** Explain carefully how and why the check column works. Use the properties of elementary row operations and the solutions of linear systems.

Solution:

For number 2, 0/4 if no explanation is given for the answer, i.e. no work is shown.
For number 3, 0/4 if the solution (the RREF) is wrong. Since a check column is being used, there should be no chance of getting the arithmetic wrong. It is interesting to note that there were very few exam papers where the check column was not used that had the correct answer. And, there were very few papers where the check column was used correctly that had the wrong RREF.
For the bonus number 4, one needs to provide an explanation using the fact that the solution set of a linear system does not change under elementary row operations and the vector of ones is a solution for the check column.

A MATLAB program for the solution is:

```
clear all
!rm output
diary output
disp('coefficient matrix is ')
A= [ 1    i     1-2*i
   2      -1+2*i      1-5*i
   -1      0      -2+3*i  ]
disp('augmented matrix is ')
Aug=[A [-i;  2-2i;  -i] ]
disp('augmented matrix with check column is ')
Augc=[Aug Aug*ones(4,1)]
disp('augmented matrix with wrong check column is ')
Augerror=[Aug [4; -i; 5]]
Augerror(2,:)= Augerror(2,:)-2*Augerror(1,:)
Augerror(3,:)= Augerror(3,:)+Augerror(1,:)
disp('one gets the wrong pivoted check column')
Augerror(:,5)
Augc(2,:)= Augc(2,:)-2*Augc(1,:)
Augc(3,:)= Augc(3,:)+Augc(1,:)
```

```
Augc(1,:)= Augc(1,:)+i*Augc(2,:)
Augc(3,:)= Augc(3,:)+i*Augc(2,:)
Augc(2,:)= -Augc(2,:)
disp('a particular solution is : ')
xp=[Augc(1:2,4); 0]
disp('a solution to the homogeneous equation is : ')
xh=[Augc(1:2,3); -1]
disp(' the general solution with parameter v is     xp + v xh ')
diary off
```

The output from the matlab program is

```
coefficient matrix is

A =

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i
   2.0000            -1.0000 + 2.0000i   1.0000 - 5.0000i
  -1.0000                  0            -2.0000 + 3.0000i

augmented matrix is

Aug =

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
   2.0000            -1.0000 + 2.0000i   1.0000 - 5.0000i   2.0000 - 2.0000i
  -1.0000                  0            -2.0000 + 3.0000i        0 - 1.0000i

augmented matrix with check column is

Augc =

  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
   2.0000            -1.0000 + 2.0000i   1.0000 - 5.0000i   2.0000 - 2.0000i
  -1.0000                  0            -2.0000 + 3.0000i        0 - 1.0000i

  Column 5

   2.0000 - 2.0000i
   4.0000 - 5.0000i
  -3.0000 + 2.0000i

augmented matrix with wrong check column is

Augerror =
```

```
  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
   2.0000            -1.0000 + 2.0000i   1.0000 - 5.0000i   2.0000 - 2.0000i
  -1.0000                  0            -2.0000 + 3.0000i        0 - 1.0000i

  Column 5

   4.0000
        0 - 1.0000i
   5.0000


Augerror =

  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
  -1.0000                  0            -2.0000 + 3.0000i        0 - 1.0000i

  Column 5

   4.0000
  -8.0000 - 1.0000i
   5.0000


Augerror =

  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
        0                  0 + 1.0000i  -1.0000 + 1.0000i        0 - 2.0000i

  Column 5

   4.0000
  -8.0000 - 1.0000i
   9.0000

one gets the wrong pivoted check column

ans =
```

```
   4.0000
  -8.0000 - 1.0000i
   9.0000


Augc =

  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
  -1.0000                  0            -2.0000 + 3.0000i        0 - 1.0000i

  Column 5

   2.0000 - 2.0000i
        0 - 1.0000i
  -3.0000 + 2.0000i


Augc =

  Columns 1 through 4

   1.0000                  0 + 1.0000i   1.0000 - 2.0000i        0 - 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
        0                  0 + 1.0000i  -1.0000 + 1.0000i        0 - 2.0000i

  Column 5

   2.0000 - 2.0000i
        0 - 1.0000i
  -1.0000


Augc =

  Columns 1 through 4

   1.0000                  0             2.0000 - 3.0000i        0 + 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
        0                  0 + 1.0000i  -1.0000 + 1.0000i        0 - 2.0000i

  Column 5
```

```
   3.0000 - 2.0000i
        0 - 1.0000i
  -1.0000


Augc =

  Columns 1 through 4

   1.0000                  0             2.0000 - 3.0000i        0 + 1.0000i
        0            -1.0000            -1.0000 - 1.0000i   2.0000
        0                  0                  0                  0

  Column 5

   3.0000 - 2.0000i
        0 - 1.0000i
        0


Augc =

  Columns 1 through 4

   1.0000                  0             2.0000 - 3.0000i        0 + 1.0000i
        0             1.0000             1.0000 + 1.0000i  -2.0000
        0                  0                  0                  0

  Column 5

   3.0000 - 2.0000i
        0 + 1.0000i
        0

a particular solution is :

xp =

        0 + 1.0000i
  -2.0000
        0

a solution to the homogeneous equation is :

xh =
```

```
  2.0000 - 3.0000i
  1.0000 + 1.0000i
 -1.0000

the general solution with parameter v is     xp + v xh
```