1. i) State the definition of the Laplace transform $\mathcal{L}(f(t))$ and use it to find $\mathcal{L}(1)$

The Laplace transform is defined as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{L\to\infty} \int_0^L e^{-st} f(t)\,dt$$

Thus

$$\mathcal{L}(1) = \int_0^\infty 1 e^{-st}\,dt = \lim_{L\to\infty} \int_0^L e^{-st}\,dt = \lim_{L\to\infty} \left(\frac{e^{-st}}{-s}\Big|_0^L \right) = \lim_{L\to\infty} \left[\frac{1}{s} - \frac{e^{-sL}}{s} \right]$$

$= \frac{1}{s}$ since $\lim_{L\to\infty} e^{-sL} = 0$ for all $s > 0$

ii) If $F(s) = \mathcal{L}(f(t))$ use definition of the Laplace transform to find $\mathcal{L}(\frac{df}{dt})$.

By definition $\mathcal{L}\left(\frac{df}{dt}\right) = \lim_{L\to\infty} \int_0^L \frac{df}{dt} e^{-st}\,dt$

Now integrate by parts

$$= \lim_{L\to\infty} \left[f(t)e^{-st}\Big|_0^L - \int_0^L f(t) \frac{de^{-st}}{dt}\,dt \right]$$

but $\lim_{L\to\infty} f(L)e^{-sL} = 0$

$$= \lim_{L\to\infty} \left[f(L)e^{-sL} - f(0) + s\int_0^L f(t)e^{-st}\,dt \right]$$

thus

$$= s \lim_{L\to\infty} \int_0^L f(t)e^{-st}\,dt - f(0) = s\mathcal{L}(f(t)) - f(0)$$

iii) Use Laplace transforms (and the given table) to solve the differential equation

$$\frac{dy}{dt} + y = 17,$$

with the initial conditions $y(0) = -1$.

Take Laplace transform & use $\mathcal{L}\left(\frac{dy}{dt}\right) = sY(s) - y(0)$ to get

$$sY(s) - y(0) + Y(s) = \frac{17}{s}$$

rearrange & use $y(0) = -1$

$$(s+1)Y(s) = \frac{17}{s} - 1$$

Thus

$$Y(s) = \frac{17}{s(s+1)} - \frac{1}{s+1}$$

; Now use partial fractions

$$\frac{17}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

means $As + A + Bs = 17$

$\therefore A = 17$

$B = -A = -17$

$$= \frac{17}{s} - \frac{17}{s+1} - \frac{1}{s+1}$$

$$Y(s) = \frac{17}{s} - \frac{18}{s+1}$$

and from tables

$$y(t) = 17 - 18e^{-t}$$

check

$$\frac{dy}{dt} = +18e^{-t}$$

$\therefore \frac{dy}{dt} + y = 17$ ✓

2. i) Solve the differential equation

$$\frac{dy}{dt} - y(t)^2 t = 0$$

with the initial condition $y(0) = 1/2$. For what values of t is the solution valid?

The DE is separable $\frac{dy}{dt} = y(t)^2 t$

$\frac{dy}{y^2} = t\,dt$

integrate to get $-\frac{1}{y} = \frac{t^2}{2} + C$ but $y(0) = \frac{1}{2}$

so that $C = -2$ and $\frac{1}{y(t)} = 2 - \frac{t^2}{2} = \frac{4-t^2}{2}$

Thus $y(t) = \frac{2}{4-t^2}$ and the solution is valid for $0 \le t < 2$

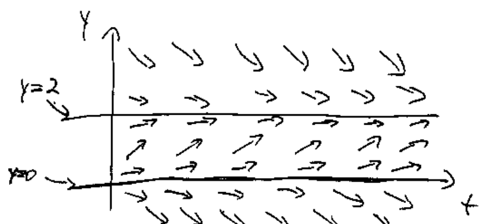
ii) Sketch the direction field and find all fixed points (or equilibrium points) for the differential equation

$$\frac{dy}{dt} + y(t)(y(t) - 2) = 0 \quad \text{or } \frac{dy}{dt} = y(2-y)$$

at fixed points $\frac{dy}{dt} = 0$, this happens when $y = 0$ or $y = 2$

$\frac{dy}{dt} > 0$ when $0 < y < 2$

$\frac{dy}{dt} < 0$ when $y < 0$ or $y > 2$



iii) Without solving the above differential equation find the limit of $y(t)$ as $t \longrightarrow \infty$ of the above equation for the two initial conditions $y_1(0) = 1$ and $y_2(0) = -1$.

when $y_1(0) = 1$ we have $\frac{dy}{dt} > 0$

since $y_1(t)$ cannot grow past the fixed point at $y = 2$ we must have $y(t) \to 2$ as $t \to \infty$

when $y_2(0) = -1$ we have $\frac{dy}{dt} < 0$ so $y_2(t)$ is decreasing. Since the more negative y is the larger the magnitude of $y(2-y)$ is we must have $y_2(t) \to -\infty$ as $t \to \infty$

3. A ball of mass m is dropped from the CN tower and falls under the action of the force of gravity $F_g = -mg$ and air resistance $F_r = -kv(t)$. The position is given by $x(t)$ and the origin of the coordinate system is assumed to match the ball's position at $t = 0$. The ball is not moving at $t = 0$.

i) From the sign of the two forces, which way does the axis point?

F_g points downward, toward the center of the Earth, thus

and since $v(0)=0$ and the ball falls downward $v(t)<0$. Hence $F_r = -kv(t) > 0$ and friction acts upwards against the direction of motion

ii) Write down Newton's law for this situation and rewrite it as a differential equation for $v(t)$ NOT $x(t)$. What is the initial condition?

$$ma = m\frac{dv}{dt} = F_g + F_r = -mg - kv$$

$$v(0)=0$$

Let $g=10$ $k=1$ $m=1$.

iii) By solving the differential equation OR by some other means show that the ball reaches a terminal velocity and find an expression for the terminal velocity.

$g=10$, $k=m=1$ means

$$\begin{cases} \dfrac{dv}{dt} = -10 - v \\ v(0)=0 \end{cases}$$

either

$$s\bar{V}(s) = -\frac{10}{s} - \bar{V}(s)$$

or $\bar{V}(s)(s+1) = -\frac{10}{s}$

$$\bar{V}(s) = \frac{-10}{s(s+1)} = \frac{-10}{s} + \frac{10}{s+1}$$

$\therefore$ $v(t) = -10 + 10e^{-t} = -10(1-e^{-t})$ $\quad$ $\therefore$ $v(t) \to -10$ as $t \to \infty$

OR

Notice $\frac{dv}{dt}=0$ when $-10-v=0$ or $v=-10$

Now when $v>-10$ $\frac{dv}{dt}<0$

$\therefore$ $v(t) \to -10$ as $t\to\infty$

4. i) A coffee mug of height 0.2 meters has a circular cross-section that varies in radius with height as

$$r(z) = 0.05 + 0.05\frac{z}{0.2}$$

where $0 \leq z \leq 0.2$. Make a careful sketch of the mug and find the volume of coffee it contains when full.

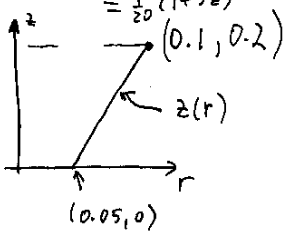
ii) Given that he concentration of caffeine varies as a function of height according to

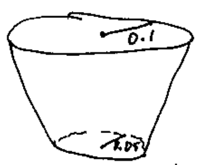
$$C(z) = C_0(1 + \frac{z}{0.2})$$

find the average concentration of caffeine in the cup (your answer will depend on C_0).

i) $r(z) = \frac{1}{20} + \frac{1}{4}z$

$= \frac{1}{20}(1+5z)$





note: $r(z)^2 = \frac{1}{400} + \frac{z}{40} + \frac{z^2}{16}$

5

Since r is given as r(z) use the cylinder technique

$$V = \int_{z=0}^{z=0.2} \pi\, r(z)^2\, dz$$

$$= \int_0^{0.2} \frac{\pi}{20^2}\left(1+10z+25z^2\right) dz$$

$$= \frac{\pi}{400}\left(z + 5z^2 + \frac{25}{3}z^3\right)\Big|_0^{1/5}$$

$$= \frac{\pi}{400}\left[\frac{1}{5} + \frac{5}{25} + \frac{25}{3}\frac{1}{125}\right]$$

$$= \frac{7}{6000}\pi \approx 0.00367\, m^3$$

ii) $\langle C \rangle$ will denote the average

By definition $\langle C \rangle = \frac{\text{total } C}{\text{volume}}$

Let $C_T \equiv$ total C

$$C_T = \int_{z=0}^{z=0.2} \pi\, r(z)^2\, C(z)\, dz = \int_0^{0.2} \pi\left(\frac{1}{400} + \frac{z}{40} + \frac{z^2}{16}\right)(1+5z)C_0\, dz$$

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$$= \pi C_0 \int_0^{0.2}\left(\frac{1}{400} + \frac{3}{80}z + \frac{3}{16}z^2 + \frac{5}{16}z^3\right) dz$$

$$= \pi C_0\left[\frac{1}{400}z + \frac{3}{80}\frac{z^2}{2} + \frac{z^3}{16} + \frac{5 \cdot z^4}{64}\right]\Big|_{z=0}^{z=\frac{1}{5}}$$

$$= \pi C_0\left[\frac{1}{2000} + \frac{3}{4000} + \frac{1}{2000} + \frac{5}{40\,000}\right] = \frac{3}{1600}\pi C_0$$

$$\therefore \langle C \rangle = \frac{45}{28}C_0 \approx 1.61\, C_0$$

5. i) Does the integral

$$\int_1^\infty \exp(-s^2)ds$$

converge? Justify your answer by a suitable Comparison Theorem.

Consider $\exp\{-\square\}$. We know this function is decreasing for all inputs. Moreover for all $1 \le s < \infty$ $s^2 \ge s$. Thus for all $1 \le s < \infty$ $|\exp\{-s^2\}| \le |\exp\{-s\}|$

(5)

Next note $\int_1^\infty e^{-s}ds = \lim_{L\to\infty} \int_1^L e^{-s}ds = \lim_{L\to\infty} -e^{-s}\Big|_1^L = \lim_{L\to\infty}(\frac{1}{e}-e^{-L}) = \frac{1}{e}$

By the Comparison Theorem $\int_1^\infty \exp\{-s^2\}ds$ converges.

ii) If the position of a particle is given by

$$x(t) = \int_0^{t^2} \sin(s^3)ds$$

Find the velocity $v(t) = \frac{dx}{dt}$ and acceleration $a(t) = \frac{d^2x}{dt^2}$.

By FTC II we have $x(t) = F(t^2) - \underbrace{F(0)}_{\text{constant}}$ where $F'(s) = \sin(s^3)$

$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}F(t^2) - \frac{d}{dt}F(0) = \frac{dF(\square)}{d\square}\frac{d\square}{dt} - 0$ by Chain Rule

(5) where $\square = t^2$ so that $\frac{d\square}{dt} = 2t$ and $\frac{dF(\square)}{d\square} = \sin(\square^3)$

$\therefore$ $v(t) = \sin[(t^2)^3]2t = 2t\sin(t^6)$. Next $a(t) = \frac{dv}{dt}$

by product & chain rule $\frac{dv}{dt} = 2\sin(t^6) + 2t\cos(t^6)6t^5$

$\therefore$ $a(t) = 2\sin(t^6) + 12t^6\cos(t^6)$

iii) If the velocity is given by

$$v(t) = \frac{t}{1+t^4}$$

and $x(0) = 2$ find $x(t)$ and evaluate both $x(1)$ and the limit of $x(t)$ as $t \longrightarrow \infty$.
HINT: recall that

$$\arctan(s) = \int \frac{1}{1+s^2}ds$$

as $t \to \infty$ $\arctan(t^2) = \frac{\pi}{2}$ so $x(t) \to \frac{\pi}{4}+2$

We know $v(t) = \frac{dx}{dt}$ so to get $x(t)$ we integrate $v(t)$

(5)

$\int v(t)\,dt = \int \frac{t}{1+t^4}dt$ Using the hint let $u = t^2$ $\frac{du}{dt} = 2t$ or $dt = \frac{du}{2t}$

$= \int \frac{t}{1+u^2}\frac{du}{2t} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan(u) + C$

$= \frac{1}{2}\arctan(t^2) + C$ $\boxed{x(1) = \frac{\pi}{8}+2}$

so $x(t) = \frac{1}{2}\arctan(t^2) + C$ but $x(0) = 2$ so $2 = \frac{1}{2}\arctan(0) + C$

since $\arctan(0) = 0$ we get $C = 2$ and $x(t) = \frac{1}{2}\arctan(t^2) + 2$

6. The velocity of a forced simple harmonic oscillator is given by the equation

$$v(t) = t\cos(t)$$

i) Sketch $v(t)$, find $v(0)$ and discuss the behaviour of $v(t)$ for large t.

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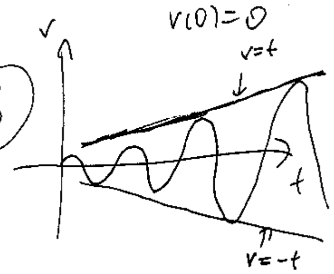
$v(0) = 0$



$$v(t) = \underset{\text{envelope}}{t}\ \underbrace{\cos t}_{\text{oscillation}}$$

as $t \to \infty$ the envelope gives the amplitude of the oscillations.

In the present case the envelope grows without bound and hence so does the amplitude of the oscillations

ii) If $x(0) = -1$ use the $v(t)$ from part i) to find $x(t)$. Discuss the behaviour of $x(t)$ for large t.

$$x(t) = \int v(t)\,dt = \int t\cos t\,dt$$

integrate by parts $= t\sin t - \int 1\sin t\,dt$

$$= t\sin t - \int \sin t\,dt = t\sin t + \cos t + C$$

5

$x(0) = -1$ means

$$-1 = 0 + \cos(0) + C \quad \text{or} \quad -1 = 1 + C$$

$$\therefore C = -2 \quad \text{and} \quad x(t) = \underset{\text{constant}}{-2} + \underset{\text{oscillation with constant amplitude}}{\cos t} + \underbrace{t\sin t}_{\text{oscillation with growing amplitude}}$$

as $t \to \infty$ the $t\sin t$ term dominates