MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 7 SOLUTIONS

Due **Wednesday, November 8** in Drop Box 9 before class

To receive full marks, correct answers must be fully justified.

To assist you, possible convergence tests are noted using the following abbreviations:

Divergence Test=DT, Integral Test=IT, Comparison Test=CT,
Limit Comparison Test=LCT, Ratio Test=RT, Alternating Series Test=AST.

Not all valid tests/methods are necessarily listed for each problem.
State which test you use and explain all steps.

1. Determine whether the following series converge or diverge.

   (a) $\sum_{n=1}^{\infty} \frac{n+e}{n^2}$ (LCT, CT, or expand)

   Solution (LCT): Since $a_n = \frac{n+e}{n^2} \approx \frac{n}{n^2} = \frac{1}{n}$ for large n, let $b_n = \frac{1}{n}$. Since $\frac{a_n}{b_n} = \frac{\frac{n+e}{n^2}}{\frac{1}{n}} = \frac{n+e}{n} = 1 + \frac{e}{n} \to 1 > 0$ as $n \to \infty$, and since $\sum b_n$ is a divergent p-series ($p = 1$), then $\sum a_n$ is also divergent, by the LCT.

   Solution (CT): Since $n + e > n \Rightarrow \frac{n+e}{n^2} > \frac{n}{n^2} = \frac{1}{n}$, and since $\sum \frac{1}{n}$ is a divergent p-series (p=1), then $\sum \frac{n+e}{n^2}$ diverges by the CT.

   Solution (expand): Since $\sum \frac{n+e}{n^2} = \sum \frac{1}{n} + \frac{e}{n^2} = \sum \frac{1}{n} + \sum \frac{e}{n^2}$ is the sum of a divergent p-series (p=1) and a convergent p-series (p=2), respectively, the given series is divergent.

   (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ (IT)

   Solution (IT): Let $f(x) = \frac{1}{x(\ln x)^3}$ so that $a_n = f(n)$. For $x \geq 2$, $f(x)$ is positive and continuous. We can see that $f(x)$ must be decreasing since, as x increases, $\ln x$ increases, so then must the product $x(\ln x)^3$ increase. Therefore, the reciprocal $1/(x(\ln x)^3)$ must decrease. Or, we can prove $f(x)$ is decreasing by examining the derivative of $f(x)$:

   $$f'(x) = \frac{d}{dx}(x(\ln x)^3)^{-1} = -\left[x((\ln x)^3)^{-2} \cdot \left((\ln x)^3 + 3x(\ln x)^2 \cdot \frac{1}{x}\right)\right] < 0$$

   (all terms inside the square brackets [ ] are positive).

   With the substitution $u = \ln x \Rightarrow du = \frac{dx}{x}$ we have:

   $$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^{\infty} \frac{du}{u^3} = \lim_{b\to\infty} \left[-\frac{1}{2u^2}\right]_{\ln 2}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}$$

   Since $\int_2^{\infty} f(x)\,dx$ converges, then $\sum_{n=2}^{\infty} a_n$ also converges, by the IT.

(c) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ (DT or RT)

Solution (DT): Since $\frac{n!}{3^n} \to \infty$ as $n \to \infty$ (see Assignment 6), then $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ diverges by the DT.

Solution (RT): Let $a_n = \frac{n!}{3^n}$. Then $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \frac{(n+1)!3^n}{3^{n+1}n!} = \frac{n+1}{3} \to \infty > 1$ as $n \to \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by the RT.

(d) $\sum_{n=1}^{\infty} \frac{1}{4^n+4n}$ (CT or RT)

Solution (CT): Since $4^n + 4n > 4^n \Rightarrow \frac{1}{4^n+4n} < \frac{1}{4^n}$, and since $\sum \frac{1}{4^n} = \sum \left(\frac{1}{4}\right)^n$ is a convergent geometric series ($r = \frac{1}{4} < 1$), then $\sum_{n=1}^{\infty} \frac{1}{4^n+4n}$ converges by the CT.

Solution (RT): Let $a_n = \frac{1}{4^n+4n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{4^{n+1}+4(n+1)}}{\frac{1}{4^n+4n}} = \frac{4^n + 4n}{4^{n+1} + 4(n+1)} = \frac{4^n + 4n}{4(4^n + n + 1)} \frac{\frac{1}{4^n}}{\frac{1}{4^n}} = \frac{1 + \frac{4n}{4^n}}{4\left(1 + \frac{4n}{4^n} + \frac{1}{4^n}\right)} \to \frac{1}{4} < 1 \text{ as n} \to \infty,$$

so $\sum_{n=1}^{\infty} a_n$ converges by the RT.

(e) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ (RT)

Solution (RT): Let $a_n = \frac{3^n}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \frac{3^{n+1}n!}{(n+1)!3^n} = \frac{3}{n+1} \to 0 < 1$ as $n \to \infty$, so $\sum_{n=1}^{\infty} a_n$ converges by the RT.

(f) $\sum_{n=1}^{\infty} \frac{\sin n}{e^n}$ (CT)

Solution (CT): Since $|\sin n| \leq 1$ for all $n \geq 1$, then $|a_n| = \left|\frac{\sin n}{e^n}\right| \leq \frac{1}{e^n}$, and since $\sum \frac{1}{e^n} = \sum \left(\frac{1}{e}\right)^n$ is a convergent geometric series ($r = 1/e < 1$), then $\sum_{n=1}^{\infty} \left|\frac{\sin n}{e^n}\right|$ converges by the CT. Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent.

(g) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ (CT or IT)

Solution (CT): For $n \geq 3$, $\ln n > 1 \Rightarrow (\ln n)^2 > 1^2 = 1 \Rightarrow \frac{(\ln n)^2}{n} > \frac{1}{n}$, and since $\sum_{n=3}^{\infty} \frac{1}{n}$ is a divergent p-series (p=1), $\sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$ also diverges, by the CT. Thus $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} = \sum_{n=1}^{2} \frac{(\ln n)^2}{n} + \sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$ also diverges.

Solution (IT): Let $f(x) = \frac{(\ln x)^2}{x}$ so that $a_n = f(n)$. For $x \in (1, \infty)$, $f(x)$ is positive and continuous. We consider the derivative of $f(x)$ to determine whether $f(x)$ is decreasing:

$$f'(x) = \frac{(2(\ln x) \cdot \frac{1}{x})x - (\ln x)^2 \cdot 1}{x^2} = \frac{\ln x(2 - \ln x)}{x^2} < 0 \Leftrightarrow 2 - \ln x < 0 \Rightarrow 2 < \ln x \Rightarrow x > e^2.$$

Thus, the IT can be used for $x > e^2$. We must choose the next highest integer value k for x (to correspond to the index n of the series):

- Without a calculator: since $2 < e < 3 \Rightarrow e^2 < 9 = k$.
- With a calculator: since $e^2 \approx 7.4 \Rightarrow k = 8$.

Hence we apply the IT for $x \in [k, \infty)$ to draw conclusions for the corresponding series $\sum_{n=k}^{\infty} \frac{(\ln n)^2}{n}$.

$\Big\{$ Note: since $\sum_{n=1}^{k} \frac{(\ln n)^2}{n}$ is finite, our IT conclusion regarding $\sum_{n=k}^{\infty} \frac{(\ln n)^2}{n}$ will also apply to $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$. $\Big\}$

With the substitution $u = \ln x \Rightarrow du = \frac{dx}{x}$ and for $k = 8$ or 9, we have:

$$\int_k^{\infty} f(x)\,dx = \int_k^{\infty} \frac{(\ln x)^2}{x}\,dx = \int_{\ln k}^{\infty} u^2\,du = \lim_{b\to\infty} \left[\frac{u^3}{3}\right]_{\ln k}^{b} = \lim_{b\to\infty} \left(\frac{b^3}{3} - \frac{(\ln k)^3}{3}\right) \to \infty.$$

Since $\int_k^{\infty} f(x)\,dx$ diverges, so does $\sum_{n=k}^{\infty} \frac{(\ln n)^2}{n}$, by the IT. Therefore, $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ also diverges.

(h) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (CT or RT)

Solution (CT): Since

$$a_n = \frac{n!}{n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}{n \cdot n \cdot n \cdots n \cdot n} \le \frac{1}{n} \cdot \frac{2}{n} \cdot 1 \cdot 1 \cdots 1 = \frac{2}{n^2} \text{ for n} \ge 2$$

(since $\frac{3}{n}, \frac{4}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}$ are each less than or equal to 1 when $n \ge 2$) and since $\sum \frac{2}{n^2}$ is a convergent p-series (p=2), then $\sum_{n=2}^{\infty} \frac{n!}{n^n}$ converges by the CT. Therefore, $\sum_{n=1}^{\infty} \frac{n!}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{n!}{n^n}$ is also convergent.

Solution (RT): Let $a_n = \frac{n!}{n^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n!n^n}{(n+1)(n+1)^n n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$$

as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n$ converges by the RT.

2. Determine whether the following series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ (AST and LCT)

Solution (AST + LCT): Let $a_n = \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$. Then $|a_n| = \frac{1}{\sqrt{n(n+1)}}$. Since $n(n+1)$ is increasing for $n \ge 1$ and since $\sqrt{\cdot}$ is an increasing function, then the sequence $\{|a_n|\}$ decreases to 0 as $n \to \infty$. Thus, $\sum a_n$ converges by the AST. Since $|a_n| = \frac{1}{\sqrt{n(n+1)}} \approx \frac{1}{\sqrt{n^2}} = \frac{1}{n}$, let $b_n = \frac{1}{n}$. Since

$$\frac{|a_n|}{b_n} = \frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{\sqrt{\frac{n^2+n}{n^2}}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \to 1 > 0 \text{ as n} \to \infty,$$

and $\sum b_n$ is a divergent p-series (p=1), then $\sum |a_n|$ diverges by the LCT. Thus, $\sum a_n$ converges conditionally.

(b) $\sum_{n=1}^{\infty} \frac{(-5)^n}{4^{2n+1}}$ (AST or RT)

Solution (absolute convergence): Let $a_n = \frac{(-5)^n}{4^{2n+1}} = \frac{(-1)^n 5^n}{4(16)^n} = \frac{(-1)^n}{4}\left(\frac{5}{16}\right)^n$. Then $|a_n| = \frac{1}{4}\left(\frac{5}{16}\right)^n$. Since $\sum |a_n|$ is a convergent geometric series ($r = 5/16 < 1$), then $\sum a_n$ converges absolutely.

Solution (AST): Let $a_n = \frac{(-5)^n}{4^{2n+1}} = \frac{(-1)^n 5^n}{4(16)^n} = \frac{(-1)^n}{4}\left(\frac{5}{16}\right)^n$. Then $|a_n| = \frac{1}{4}\left(\frac{5}{16}\right)^n$. Since the sequence $\{|a_n|\}$ decreases to 0 as $n \to \infty$, then $\sum a_n$ converges by the AST. Since $\sum |a_n|$ is a convergent geometric series ($r = 5/16 < 1$), then $\sum a_n$ converges absolutely.

Solution (RT): Let $a_n = \frac{(-5)^n}{4^{2n+1}} = \frac{(-1)^n 5^n}{4(16)^n} = \frac{(-1)^n}{4}\left(\frac{5}{16}\right)^n$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}}{4}\left(\frac{5}{16}\right)^{n+1}}{\frac{(-1)^n}{4}\left(\frac{5}{16}\right)^n}\right| = \frac{5}{16} < 1$, hence $\sum a_n$ converges absolutely by the RT.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt[4]{n}}$ (AST)

Solution (AST): Let $a_n = \frac{(-1)^n}{2\sqrt[4]{n}}$. Since the sequence $\{|a_n|\}$ decreases to 0 as $n \to \infty$, then $\sum a_n$ converges by the AST. The series $\sum |a_n|$ is a divergent p-series (p=1/4), hence $\sum a_n$ converges conditionally.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$ (DT or RT)

$\left\{\right.$ Note: AST does not apply, since the terms of the sequence $\left\{\left|\frac{(-1)^n n!}{3^n}\right|\right\}$ do not tend to 0 as $n \to \infty$. $\left.\right\}$

Solution (DT): Let $a_n = \frac{(-1)^n n!}{3^n}$. Then $|a_n| \to \infty$ as $n \to \infty$ (see #1(c)). Since $|a_n|$ does not tend to 0, then a_n does not tend to 0. Thus, $\sum a_n$ diverges by the DT.

Solution (RT): Let $a_n = \frac{(-1)^n n!}{3^n}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)!}{3^{n+1}}}{\frac{(-1)^n n!}{3^n}}\right| = \left|\frac{(-1)^{n+1}(n+1)!3^n}{(-1)^n n! 3^{n+1}}\right| = \frac{n+1}{3} \to \infty$ as $n \to \infty$. Therefore, $\sum a_n$ diverges by the RT.

Bonus Question

Suppose you are offered the choice between the following four deals, where you get the following specific dollar amounts on the day indicated. Assume the deal lasts forever.

Deal #	Day 1	Day 2	Day 3	Day 4	...
1	\$1	\$0.50	$\frac{1}{4}$ of \$1	$\frac{1}{8}$ of \$1	...
2	\$1	\$0.50	$\frac{1}{3}$ of \$1	$\frac{1}{4}$ of \$1	...
3	\$1	\$0.50	$\frac{1}{6}$ of \$1	$\frac{1}{24}$ of \$1	...
4	\$1	\$0.25	$\frac{1}{9}$ of \$1	$\frac{1}{16}$ of \$1	...

Which deal should you choose to ensure you receive an infinite amount of money? Justify your answer.

Solution:

Deal 1: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$, a convergent geometric series (r=1/2) whose sum is $\frac{1}{1-\frac{1}{2}} = 2$. So, Deal 1 yields a maximum of \$2.

Deal 2: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series (p=1). So, Deal 2 yields an infinite amount of money.

Deal 3: $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + \frac{1}{n!} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n!}$. We can determine the convergence of this series using the RT. Let $a_n = \frac{1}{n!}$. Then $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 < 1$ as $n \to \infty$. Thus, $\sum \frac{1}{n!}$ is convergent, by the RT. So, Deal 3 yields a finite amount of money.

[Actually, the value of the related series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is known to be e, derived from the Maclaurin series for e^x when $x = 1$. See Section 8.7 of the textbook. So, $\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{1} \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} = (e-1)$.]

Deal 4: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series (p=2). So, Deal 4 yields a finite amount of money.

[Actually, we can tell from the IT that the value for the series must be less than 2. (See pages 577-578 of the textbook.) A complicated proof, using power series (Sections 8.5 and 8.6 of the textbook), and beyond the scope of this course, reveals the actual value is $\pi^2/6$.]

Conclusion: Select Deal 2, since it will generate an infinite amount of money.