MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 7

1: Determine which of the following series converge.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{n^2+4n}{\sqrt{n^5-2n+1}}$

Solution: Let $a_n=\dfrac{n^2+4n}{\sqrt{n^5-2n+1}}$ and let $b_n=\dfrac{n^2}{\sqrt{n^5}}=\dfrac{1}{\sqrt{n}}$. Then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1+\frac{4}{n}}{\sqrt{1-\frac{2}{n^4}+\frac{1}{n^5}}}=1$, and $\sum b_n$ diverges (its a p-series with $p=\frac{1}{2}$), and so $\sum a_n$ diverges too, by the Limit Comparison Test.

(b) $\displaystyle\sum_{n=1}^{\infty}e^{1/n}$

Solution: Since $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$, we have $\displaystyle\lim_{n\to\infty}e^{1/n}=e^0=1$, and so $\sum e^{1/n}$ diverges by the Divergence Test.

2: Determine which of the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: Let $a_n=\dfrac{n!}{n^n}$. Note that $0\le a_n=\frac{1\cdot2\cdot3\cdots n}{n\cdot n\cdot n\cdots n}=\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\le\frac{1}{n}\cdot\frac{2}{n}=\frac{2}{n^2}$ for $n\ge2$. We know that $\sum\frac{2}{n^2}$ converges (its a constant times a p-series), so $\sum a_n$ converges too, by the Comparison Test.

Here is second solution. We have $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\,\frac{n^n}{n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}e^{n\ln\left(\frac{n}{n+1}\right)}$. By l'Hôpital's Rule, $\displaystyle\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{n+1}{n}\,\frac{n+1-n}{(n+1)^2}}{-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{-n}{n+1}=-1$, and so $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=e^{-1}<1$. Thus, by the Ratio Test, $\sum a_n$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$

Solution: Let $a_n=\dfrac{\ln n}{n^2}$. Since $\ln n<\sqrt{n}$ for all $n\ge1$, we have $0<a_n<\dfrac{\sqrt{n}}{n^2}=\dfrac{1}{n^{3/2}}$, and we know that $\displaystyle\sum\frac{1}{n^{3/2}}$ converges (its a p-series), and so $\sum a_n$ converges too, by the Comparison Test. (We can prove that $\ln x<\sqrt{x}$ for all $x>0$ as follows: let $f(x)=\sqrt{x}-\ln x$. Then $f'(x)=\dfrac{1}{2\sqrt{x}}-\dfrac{1}{x}=\dfrac{\sqrt{x}-2}{2x}$, so we have $f'(4)=0$, and $f'(x)<0$ when $x<4$, and $f'(x)>0$ when $x>4$, and so $f(x)$ reaches its minimum when $x=4$. But $f(4)=2-\ln4>0$, and so $f(x)>0$ for all x).

Here is a second solution. Let $f(x)=\dfrac{\ln x}{x^2}$ so that $a_n=f(n)$. Note that $f'(x)=\dfrac{x-2x\ln x}{x^4}=\dfrac{1-2\ln x}{x^3}$, so we have $f'(x)<0$ when $x>\sqrt{e}$, and so $f(x)$ is eventually decreasing and we can apply the Integral Test. Let $t=\ln x$ so that $x=e^t$ and $dt=\frac{1}{x}\,dx$. Then $\displaystyle\int_{x=1}^{\infty}f(x)\,dx=\int_{x=1}^{\infty}\frac{\ln x}{x^2}\,dx=\int_{t=0}^{\infty}t\,e^{-t}\,dt$. Integrate by parts using $u=t$ and $v=-e^{-t}$ to get $\displaystyle\int_{t=0}^{\infty}t\,e^{-t}\,dt=\Big[-t\,e^{-t}+\int e^{-t}\,dt\Big]_0^{\infty}=\Big[-t\,e^{-t}-e^{-t}\Big]_0^{\infty}=1$, since $\displaystyle\lim_{t\to\infty}t\,e^{-t}=0$ by l'Hôpital's Rule. Since the integral is finite, $\sum a_n$ converges by the Integral Test.

3: For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Solution: For $n > 1$, $\left\{\frac{1}{\ln n}\right\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, and so $\sum \frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. On the other hand, $\frac{1}{\ln n} > \frac{1}{n}$, and we know that $\sum \frac{1}{n}$ diverges, so $\sum \frac{1}{\ln n}$ diverges too, by the Comparison Test. Thus $\sum \frac{(-1)^n}{\ln n}$ is conditionally convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(-2)^n}$

Solution: Let $a_n = \dfrac{n^4}{(-2)^n}$. Then $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^4}{2^{n+1}} \frac{2^n}{n^4} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^4 = \frac{1}{2} < 1$, and so $\sum |a_n|$ converges by the Ratio Test. Thus $\sum a_n$ is absolutely convergent.

4: Let $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Find a value of l so that $S - S_l \le \frac{1}{100}$, where S_l is the l^{th} partial sum.

Solution: Let $f(x) = \dfrac{1}{x^{3/2}}$ so that $a_n = f(n)$. Note that $f(x)$ is decreasing, so we can apply the Integral Test. We have $\displaystyle S - S_l = \sum_{n=l+1}^{\infty} a_n \le \int_l^{\infty} f(x)\,dx = \int_l^{\infty} \frac{1}{x^{3/2}}\,dx = \left[-\frac{2}{x^{1/2}}\right]_l^{\infty} = \frac{2}{\sqrt{l}}$. To get $S - S_l \le \frac{1}{100}$, we can choose l so that $\frac{2}{\sqrt{l}} \le \frac{1}{100}$, that is $\sqrt{l} \ge 200$, so we can take $l = (200)^2 = 40,000$.

5: Let $S = \displaystyle\sum_{n=0}^{\infty} \frac{1}{6^n + 4}$. Find the value of a partial sum S_l such that $S - S_l \le \frac{1}{1{,}000}$.

Solution: By the Comparison Test, since $\dfrac{1}{6^n + 4} < \dfrac{1}{6^n}$, and using the formula for the sum of a geometric series, we have $\displaystyle S - S_l = \sum_{n=l+1}^{\infty} \frac{1}{6^n + 4} \le \sum_{n=l+1}^{\infty} \frac{1}{6^n} = \frac{\frac{1}{6^{l+1}}}{1 - \frac{1}{6}} = \frac{1}{5 \cdot 6^l}$. To get $S - S_l \le \frac{1}{1000}$, we can choose l so that $\dfrac{1}{5 \cdot 6^l} \le \dfrac{1}{1000}$, that is $6^l \ge 200$, so we can take $l = 3$ (since $6^3 = 216 > 200$). We have $\displaystyle S_3 = \sum_{n=0}^{3} \frac{1}{6^n + 4} = \frac{1}{5} + \frac{1}{10} + \frac{1}{40} + \frac{1}{220} = \frac{88+44+11+2}{440} = \frac{145}{440} = \frac{29}{88}$.