MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 1

1: (a) Let $f(x)=\tan^{-1}\left(\sqrt{x}\right)$. Find the second derivative $f''(1)$.

Solution: $f'(x)=\dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$ and $f''(x)=\dfrac{-1}{(1+x)^2}\cdot\dfrac{1}{2\sqrt{x}}-\dfrac{1}{1+x}\cdot\dfrac{1}{4\,x^{3/2}}$ so $f''(1)=-\frac{1}{4}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}=-\frac{1}{4}$.

(b) Find the equation of the tangent line to the curve $x^3y+3xy=y^3$ at the point $(1,-2)$.

Solution: Differentiate implicitly to get $3x^2y+x^3y'+3y+3xy'=3y^2y'$. At $(x,y)=(1,-2)$ we have $-6+y'-6+3y'=12y'$, so $8y'=-12$ hence $y'=-\frac{12}{8}=-\frac{3}{2}$. The equation of the tangent line at $(1,-2)$ is $y+2=-\frac{3}{2}(x-1)$, or equivalently $y=-\frac{3}{2}\,x-\frac{1}{2}$, or if you prefer $3x+2y+1=0$.

2: Solve the following indefinite integrals.

(a) $\displaystyle\int (x^2+1)\,e^x\;dx$

Solution: Let $u=x^2+1$ and $v=e^x$. Then $\displaystyle\int (x^2+1)\,e^x\;dx=\int u\,dv=uv-\int v\,du=(x^2+1)\,e^x-\int 2x\,e^x\;dx$.
Let $u_1=2x$ and $v_1=e^x$. Then $\displaystyle\int 2x\,e^x\;dx=\int u_1\,dv_1=u_1v_1-\int v_1\,du_1=2x\,e^x-\int 2\,e^x\;dx=2x\,e^x-2\,e^x+a$.
Thus $\displaystyle\int (x^2+1)\,e^x\;dx=(x^2+1)\,e^x-\int 2x\,e^x\;dx=(x^2+1)\,e^x-2x\,e^x+2\,e^x-a=(x^2-2x+3)\,e^x+c$

(b) $\displaystyle\int \sin^3 x\,\cos^2 x\;dx$

Solution: Let $u=\cos x$ so $du=-\sin x\;dx$. Then $\displaystyle\int \sin^3 x\cos^2 x\;dx=\int (1-\cos^2 x)\cos^2 x\;\sin x\,dx=\int -(1-u^2)\,u^2\;du=\int u^4-u^2\;du=\tfrac{1}{5}u^5-\tfrac{1}{3}u^3+c=\tfrac{1}{5}\cos^5 x-\tfrac{1}{3}\cos^3 x+c.$

3: Evaluate the following definite integrals.

(a) $\displaystyle\int_0^4 \frac{x+2}{\sqrt{2x+1}}\;dx$

Solution: Let $u=2x+1$ so $x=\frac{u-1}{2}$ and $du=2\,dx$. Then $\displaystyle\int_0^4\frac{x+2}{\sqrt{2x+1}}\;dx=\int_1^9\frac{\frac{u-1}{2}+2}{\sqrt{u}}\,\frac{1}{2}du=\frac{1}{4}\int_1^9\frac{u+3}{\sqrt{u}}\,du=\frac{1}{4}\int_1^9 u^{1/2}+3u^{-1/2}\;du=\frac{1}{4}\Big[\tfrac{2}{3}u^{3/2}+6u^{1/2}\Big]_1^9=\frac{1}{4}\Big[(18+18)-(\tfrac{2}{3}+6)\Big]=\frac{22}{3}.$

(b) $\displaystyle\int_1^2\frac{5x^2+9}{x^4-9x^2}\;dx$

Solution: We have $\dfrac{5x^2+9}{x^4-9x^2}=\dfrac{5x^2+9}{x^2(x-3)(x+3)}$. To get $\dfrac{5x^2+9}{x^4-9x^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-3}+\dfrac{D}{x+3}$, we need $A(x^3-9x)+B(x^2-9)+C(x^3+3x^2)+D(x^3-3x)=5x^2+9$. Equate coefficients to get $A+C+D=0$, $B+3C-3D=5$, $-9A=0$ and $-9B=9$. Solve these 4 equations to get $A=0$, $B=-1$, $C=1$ and $D=-1$. So $\displaystyle\int_1^2\frac{5x^2+9}{x^4-9x^2}\,dx=\int_1^2 -\frac{1}{x^2}+\frac{1}{x-3}-\frac{1}{x+3}\;dx=\Big[\tfrac{1}{x}+\ln|x-3|-\ln|x+3|\Big]_1^2=\left(\tfrac{1}{2}+\ln 1-\ln 5\right)-\left(1+\ln 2-\ln 4\right)=-\tfrac{1}{2}-\ln\tfrac{5}{2}$

4: Evaluate the following improper integrals.

(a) $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}}$

Solution: Let $u = x-9$ so $du = dx$. Then $\displaystyle\int_{x=1}^9 \frac{dx}{\sqrt[3]{x-9}} = \int_{u=-8}^0 u^{-1/3}\,du = \left[\tfrac{3}{2}\,u^{2/3}\right]_{-8}^0 = -\tfrac{3}{2}\cdot(-8)^{2/3} = -6.$

(b) $\displaystyle\int_0^2 x^3 \ln(x/2)\;dx$

Solution: Let $u = \ln(x/2)$ so $du = \frac{1}{x}\,dx$, and let $v = \frac{1}{4}x^4$ so $dv = x^3\,dx$. Then $\displaystyle\int_0^2 x^3\ln(x/2)\;dx = \int_0^2 u\,dv = \left[uv - \int v\,du\right]_0^2 = \left[\tfrac{1}{4}x^4\ln(x/2) - \int \tfrac{1}{4}x^3\,dx\right]_0^2 = \left[\tfrac{1}{4}x^4\ln(x/2) - \tfrac{1}{16}x^4\right]_0^2 = -1 - \tfrac{1}{4}\lim_{x\to 0^+} x^4\ln(x/2).$ By l'Hôpital's Rule we have $\displaystyle\lim_{x\to0^+} x^4\ln(x/2) = \lim_{x\to0^+}\frac{\ln(x/2)}{1/x^4} = \lim_{x\to0^+}\frac{1/x}{-4/x^5} = \lim_{x\to0^+} -\tfrac{1}{4}\,x^4 = 0$, and so $\displaystyle\int_0^2 x^3\ln(x/2)\;dx = -1.$

5: Evaluate the following improper integrals.

(a) $\displaystyle\int_2^\infty \frac{dx}{x^4\sqrt{x^2-4}}$

Solution: Write $\displaystyle I = \int_2^\infty \frac{dx}{x^4\sqrt{x^2-4}}$. Let $2\sec\theta = x$ so $2\tan\theta = \sqrt{x^2-4}$ and $2\sec\theta\tan\theta\,d\theta = dx$. Then $\displaystyle I = \int_0^{\pi/2}\frac{2\sec\theta\tan\theta\,d\theta}{16\sec^4\theta\,2\tan\theta} = \tfrac{1}{16}\int_0^{\pi/2}\cos^3\theta\;d\theta = \tfrac{1}{16}\int_0^{\pi/2}(1-\sin^2\theta)\,\cos\theta\,d\theta = \tfrac{1}{16}\int_0^1 1-u^2\;du$, where $u = \sin\theta$, and so $I = \frac{1}{16}\left[u - \frac{1}{3}u^3\right]_0^1 = \frac{1}{24}$.

(b) $\displaystyle\int_{-\infty}^\infty \frac{x(x+1)}{(x^2+1)^2}\;dx.$

Solution: Write $\displaystyle I = \int_{-\infty}^\infty \frac{x(x+1)}{(x^2+1)^2}$. To get $\displaystyle\frac{x(x+1)}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$ we need to have $A(x^3+x) + B(x^2+1) + Cx + D = x^2 + x$. Equate coefficients to get $A = 0$, $B = 1$, $A + C = 1$ and $B + D = 0$. Solve these to get $A = 0$, $B = 1$, $C = 1$ and $D = -1$. Thus $\displaystyle I = \int_{-\infty}^\infty \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} - \frac{1}{(x^2+1)^2}\;dx$. We have $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+1} = \left[\tan^{-1}x\right]_{-\infty}^\infty = \pi$, and $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)^2} = \left[-\frac{1}{2(x^2+1)}\right]_{-\infty}^\infty = 0$, and to get $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$ we let $\tan\theta = x$ so $\sec\theta = \sqrt{1+x^2}$ and $\sec^2\theta\,d\theta = dx$, and then $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2} = \int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta\,d\theta}{\sec^4\theta} = \int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{2}$. Thus $I = \pi + 0 - \frac{\pi}{2} = \frac{\pi}{2}$.