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H

Damped Oscillator I

Consider a damped harmonic oscillator consisting of a mass m attached to a spring with spring constant k and damping parameter b.

a) After being set in motion the position of the mass is given by $x(t) = Ae^{-\eta t}\cos(\omega t+\phi)$, where $\omega = \sqrt{\omega_\circ^2 - \eta^2}$, $\eta = b/2m$ and $\omega_\circ = \sqrt{k/m}$ (the natural frequency of the undamped oscillator). Assuming $\eta \ll \omega_\circ$, determine the mechanical energy, $E(t)$, of the oscillator as a function of time. (Hint: $E(t)$ should decay according to a simple exponential law.)

$E(t) = \frac{1}{2}kA^2e^{-2\eta t}$

b) Suppose the oscillator from part (a) is now driven with a forcing function $F(t) = F_\circ\cos(\omega_f t)$. In the steady state the position of the mass is given by $x(t) = A\cos(\omega_f t - \phi)$, where

$$A = \frac{F_\circ/m}{\sqrt{(\omega_\circ^2 - \omega_f^2)^2 + 4\eta^2\omega_f^2}}.$$

The amplitude of oscillation is maximum when the forcing frequency, ω_f, is equal to the resonance frequency, ω_r. Find this resonance frequency, ω_r, and the amplitude, A_{max}, at this frequency? (Hint: A^2 has the same maximum as A.)

$\frac{dA}{d\omega_f} = -\frac{F_\circ}{m}\frac{-\omega_f(\omega_\circ^2-\omega_(f)^2)+2\eta^2\omega_f}{((\omega_\circ^2-\omega_f^2)+4\eta^2\omega_f^2)^{3/2}}$
When A is Max $\frac{dA}{df} = 0$
$\Rightarrow 0 = -[\omega_f(\omega_\circ^2 - \omega_(f)^2) + 2\eta^2\omega_f]$
$\Rightarrow (\omega_\circ^2 - \omega_(f)^2) + 2\eta^2 = 0$
$\Rightarrow -\omega_f^2 = \omega_\circ^2 - 2\eta^2$ (when A is maximum)
$\Rightarrow \omega_r = \omega_f = \sqrt{-\omega_\circ^2 + 2\eta^2}$
$\Rightarrow A^2 = [\frac{F_\circ}{m}]^2\frac{1}{(\omega_\circ^2-\omega_r^2)^2+4\eta^2(\omega_\circ)^2}$
$\Rightarrow A^2 = \frac{F_\circ^2}{m^2 4\eta^2\omega_\circ^2}$
$\Rightarrow A_{\text{max}} = \frac{F_\circ}{2m\eta\omega_\circ}$

$\omega_r = \sqrt{-\omega_\circ^2 + 2\eta^2}$

$A_{\text{max}} = \frac{F_\circ}{2m\eta\omega_\circ}$

c) What is the $\eta \to 0$ limit of your answer to part (a)?

$\lim_{\eta\to 0} E(t) = \frac{1}{2}kA^2e^{-2\eta t}$
$= \frac{1}{2}kA^2$ (since $\mathrm{e}^0 = 1$)

$\lim_{\eta \to 0} E(t) = \frac{1}{2}kA^2$