MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 6

1: Find the limit, if it exists, of each of the following sequences.

(a) $a_n = n - \sqrt{n^2-n+2}$.

Solution: $a_n = \dfrac{\left(n-\sqrt{n^2-n+2}\right)\left(n+\sqrt{n^2-n+2}\right)}{n+\sqrt{n^2-n+2}} = \dfrac{n-2}{n+\sqrt{n^2-n+2}} = \dfrac{1-\frac{2}{n}}{1+\sqrt{1-\frac{1}{n}+\frac{2}{n^2}}} \longrightarrow \dfrac{1}{2}$.

(b) $a_n = \left(\dfrac{n+1}{n-1}\right)^n$.

Solution: $a_n = e^{n\ln\left(\frac{n+1}{n-1}\right)} \longrightarrow e^2$ since $\displaystyle\lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{n-1}{n+1}\,\frac{(n-1)-(n+1)}{(n-1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{2n^2}{n^2-1} \longrightarrow 2$, where we used l'Hôpital's Rule (treating n as a real variable).

2: Let $a_1 = 3$ and for $n \geq 1$ let $a_{n+1} = \dfrac{4}{5-a_n}$. Determine whether $\{a_n\}$ converges and, if so, find the limit.

Solution: If $\{a_n\}$ does converge with say $\displaystyle\lim_{n\to\infty} a_n = l$, then by taking the limit of both sides of the formula $a_{n+1} = \dfrac{4}{5-a_n}$ we obtain $l = \dfrac{4}{5-l}$, and we have $l = \dfrac{4}{5-l} \iff 5l - l^2 = 4 \iff l^2 - 5l + 4 = 0 \iff (l-1)(l-4) = 0 \iff l = 1$ or 4. This shows that if the limit does exist, then it must be 1 or 4. We claim that $1 \leq a_{n+1} \leq a_n \leq 4$ for all $n \geq 1$. We shall prove this claim by induction. We have $a_1 = 3$ and $a_2 = \frac{4}{5-3} = 2$, so the claim does hold when $n = 1$. Suppose that the claim holds when $n = k$ for some fixed $k \geq 1$. Then we have $1 \leq a_{k+1} \leq a_k \leq 4 \Longrightarrow -1 \geq -a_{k+1} \geq -a_k \geq -4 \Longrightarrow 4 \geq 5 - a_{k+1} \geq 5 - a_k \geq 1 \Longrightarrow \frac{1}{4} \leq \frac{1}{5-a_{k+1}} \leq \frac{1}{5-a_k} \leq 1 \Longrightarrow 1 \leq \frac{4}{5-a_{k+1}} \leq \frac{4}{5-a_k} \leq 4 \Longrightarrow 1 \leq a_{k+2} \leq a_{k+1} \leq 4$, so the claim also holds when $n = k+1$. By induction, the claim holds for all $n \geq 1$. This implies that $\{a_n\}$ is decreasing and bounded below by 1, and so it must converge. We have already seen that the limit must be 1 or 4, and since $a_1 = 3$ and $\{a_n\}$ is decreasing, the limit must be 4.

3: Let $a_1 = 2$ and for $n \geq 1$ let $a_{n+1} = \sqrt{2\,{a_n}^2 - 1}$. Determine whether $\{a_n\}$ converges and, if so, find the limit.

Solution: If $\{a_n\}$ does converge with say $\displaystyle\lim_{n\to\infty} a_n = l$ then, by taking the limit of both sides of the formula $a_{n+1} = \sqrt{2{a_n}^2-1}$, we obtain $l = \sqrt{2l^2-1}$ so $l^2 = 2l^2 - 1$ and thus $l^2 = 1$ and so $l = 1$ (note that l is positive). We claim that $1 \leq a_n \leq a_{n+1}$ for all $n \geq 1$. When $n = 1$ we have $a_n = a_1 = 2$ and $a_{n+1} = a_2 = \sqrt{7}$, so the claim holds when $n = 1$. Suppose the claim holds when $n = k$. Then $1 \leq a_k \leq a_{k+1} \Longrightarrow 1 \leq {a_k}^2 \leq {a_{k+1}}^2 \Longrightarrow 2 \leq 2{a_k}^2 \leq 2{a_{k+1}}^2 \Longrightarrow 1 \leq 2{a_k}^2 - 1 \leq 2{a_{k+1}}^2 - 1 \Longrightarrow 1 \leq \sqrt{2{a_k}^2-1} \leq \sqrt{2{a_{k+1}}^2-1} \Longrightarrow 1 \leq a_{k+1} \leq a_{k+2}$, so the claim also holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$. Since $a_1 = 2$ and $\{a_n\}$ is increasing, the limit l of $\{a_n\}$, if it exists, must be at least 2. But we already saw that if $\{a_n\}$ were to converge then its limit would have to be 1. So $\{a_n\}$ does not converge, and $\displaystyle\lim_{n\to\infty} a_n = \infty$.

4: Find the sum of each of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{2^{2n+1}}$

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{2^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} + \sum_{n=1}^{\infty} \frac{2^n}{2^{2n+1}} = \frac{\frac{1}{8}}{1-\frac{1}{4}} + \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+4n+3}$

Solution: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+4n+3} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}}{n+1} - \frac{\frac{1}{2}}{n+3}$. The l^{th} partial sum is $S_l = \frac{1}{2}\displaystyle\sum_{n=0}^{l} \frac{1}{n+1} - \frac{1}{n+3} =$ $\frac{1}{2}\Big(\big(\frac{1}{1}-\frac{1}{3}\big)+\big(\frac{1}{2}-\frac{1}{4}\big)+\big(\frac{1}{3}-\frac{1}{5}\big)+\big(\frac{1}{4}-\frac{1}{6}\big)+\cdots+\big(\frac{1}{l-2}+\frac{1}{l}\big)+\big(\frac{1}{l-1}-\frac{1}{l+1}\big)+\big(\frac{1}{l}-\frac{1}{l+2}\big)+\big(\frac{1}{l+1}-\frac{1}{l+3}\big)\Big) =$ $\frac{1}{2}\big(1+\frac{1}{2}-\frac{1}{l+2}-\frac{1}{l+3}\big) \to \frac{1}{2}\big(1+\frac{1}{2}\big) = \frac{3}{4}$ as $l \to \infty$. Thus the sum is $\frac{3}{4}$.

5: For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(5-2x^2)^n}{3^n}$ converge?

Solution: This is geometric with ratio $\dfrac{5-2x^2}{3}$, so it converges $\iff \left|\dfrac{5-2x^2}{3}\right| < 1 \iff \left|5-2x^2\right| < 3 \iff$ $|2x^2-5| < 3 \iff -3 < 2x^2-5 < 3 \iff 2 < 2x^2 < 8 \iff 1 < x^2 < 4 \iff x \in (-2,-1)\cup(1,2).$