MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 9 SOLUTIONS

Due **Friday, December 1** in Drop Box 9 before class
To receive full marks, correct answers must be fully justified.

1. (a) Find parametric equations for the line through the point $(5, 1, 0)$ that is perpendicular to the plane $2x - y + z = 1$.

   Solution: A vector normal (or perpendicular) to the plane $2x - y + z = 1$ is $\vec{n} = \langle 2, -1, 1\rangle$, which is also a direction vector for the line (since the line is also perpendicular to the plane). Thus, parametric equations for the line are

$$x = 5 + 2 \cdot t = 5 + 2t,\ \ y = 1 + (-1) \cdot t = 1 - t,\ \ z = 0 + 1 \cdot t = t$$

   (b) In what points does this line intersect the coordinate planes?

   (Hint: in each coordinate plane, the remaining variable is 0. Use this fact to solve for t and then the point in each coordinate plane.)

   Solution:

   • In the $xy-$plane, $z = 0 \Rightarrow z = t = 0 \Rightarrow x = 5, y = 1$. Therefore, the line intersects the $xy-$plane at the (given) point $(5, 1, 0)$.

   (Note: this intersection point can be deduced directly, since a line will only intersect any plane at one point, unless the line lies inside the plane—and we know our plane is not a coordinate plane.)

   • In the $yz-$plane, $x = 0 \Rightarrow 5 = -2t \Rightarrow t = -5/2 \Rightarrow y = 7/2, z = -5/2$. Therefore, the line intersects the $yz-$plane at the point $(0, 7/2, -5/2)$.

   • In the $xz-$plane, $y = 0 \Rightarrow t = 1 \Rightarrow x = 5 + 2 = 7, z = 1$. Therefore, the line intersects the $xz-$plane at the point $(7, 0, 1)$.

2. Find symmetric equations for the line of intersection of the planes $x + y - z = 2$ and $3x - 4y + 5z = -1$. (Hint: see Example 7(b), page 671.)

   Solution: Let L be the line of intersection of the two planes. We need to find a point on L and then the direction of L.

   • L will intersect at least one coordinate plane. Let $z = 0$ in the equations of both planes to yield the pair of equations $x + y = 2, 3x - 4y = -1$ whose solution is $x = 1, y = 1$. Therefore, a point on L is $(1, 1, 0)$. (Note: L intersects the $yz-$plane at the point $(0, 9, 7)$ and the $xz-$plane at the point $(9/8, 0, -7/8)$.)

   • Since L lies in both planes, it is perpendicular to each of the normal vectors of the planes. A vector $\vec{v}$ parallel to L is given by the cross product

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 3 & -4 & 5 \end{vmatrix} = \vec{i} - 8\vec{j} - 7\vec{k} \quad \Rightarrow \text{symmetric equations of } L \text{ are: } x - 1 = \frac{y-1}{-8} = \frac{z}{-7}.$$

   (Note: if a different point on L is used, then the symmetric equations will differ but they mean the same line. For example, if the point $(0, 9, 7)$ is used, the symmetric equations become $x = (y - 9)/(-8) = (z - 7)/(-7)$. Or, if the point $(9/8, 0, -7/8)$ is used, then $x - 9/8 = y/(-8) = (z + 7/8)/(-7)$.)

3. Find the distance between the parallel planes $3x + 6y - 9z = 4$ and $x + 2y - 3z = 1$.

   Solution: We need to find one point on one of the planes from which we can calculate the distance D to the other plane. Since either of the given planes must intersect two coordinate planes (since they are clearly not parallel to any coordinate plane, *i.e.*, they are not constant-valued, like the horizontal plane $z = 3$ or the vertical plane $x = -2$), we can set two variables equal to 0 in either of the planes to obtain a point in that plane.

   Let $y = z = 0$ in $3x + 6y - 9z = 4$ to obtain $x = 4/3$. Then the point $(4/3, 0, 0)$ is in the plane $3x + 6y - 9z = 4$. Then

$$D = \frac{|1 \cdot (4/3) + 2 \cdot (1) - 3 \cdot 0 - 1|}{\sqrt{1^2 + 2^2 + (-3)^2}} = \frac{1}{3\sqrt{14}}$$

4. Let $f(x, y) = \ln(x + y - 1)$.

   (a) Evaluate $f(1, 1)$.

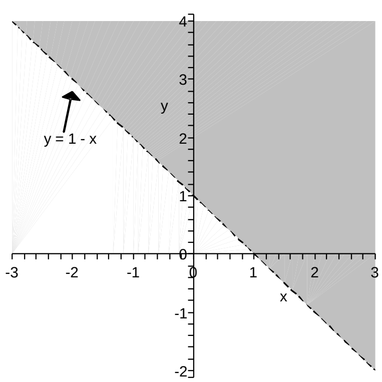
   Solution: $f(1, 1) = \ln(1 + 1 - 1) = \ln 1 = 0$.

   (b) Evaluate $f(e, 1)$.

   Solution: $f(e, 1) = \ln(e + 1 - 1) = \ln e = 1$.

   (c) Find and sketch the domain of f. (The domain lies in the xy−plane.)

   Solution: Since $\ln(x + y - 1)$ is defined only when $x + y - 1 > 0 \Rightarrow y > 1 - x$. So the domain of f is $\{(x, y) | y > 1 - x\}$, indicated by the shaded region in the figure.



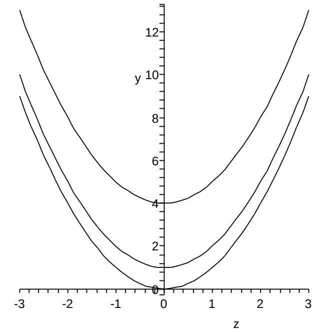
   (d) Find the range of f.

   Solution: Since $z = \ln(x + y - 1)$ can be any real number, the range is $z \in \mathbb{R}$ or $-\infty < z < \infty$.

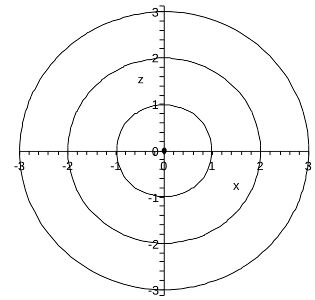
5. Use traces to sketch a graph of the surface $y = x^2 + z^2$. Follow Example 6 (page 679).
   Bonus What is the surface called?

   Solution:

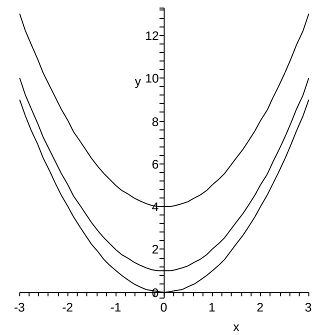
   • Traces in $x = k$ (parallel to the yz−plane) are $y = z^2 + k^2$, parabolas opening in the positive y−direction. Here we show traces for $k = 0, \pm 1, \pm 2$.



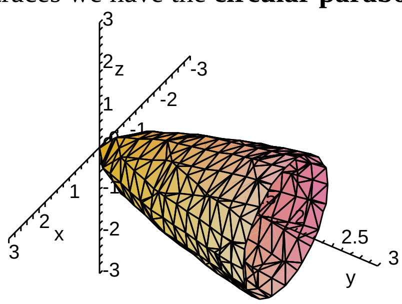
   • Traces in $y = k$ (parallel to the xz−plane) are $k = x^2 + z^2$, which reveals that $k \geq 0$. The curves are circles of radius $\sqrt{k}$, centered about the y−axis (since $x = z = 0$ there). Here we show traces for $k = 0, 1, 4, 9$.



   • Traces in $z = k$ (parallel to the xy−plane) are $y = x^2 + k^2$, parabolas opening in the positive y−direction. Here we show traces for $k = 0, \pm 1, \pm 2$.



   Combining these traces we have the **circular paraboloid**:



6. Two particles travel along the space curves $\vec{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t\rangle$.

   (a) Do the particles collide?

   Solution: The particles collide if $\vec{r}_1 = \vec{r}_2 \iff \langle t, t^2, t^3\rangle = \langle 1+2t, 1+6t, 1+14t\rangle$. Equating components gives

   $$\begin{cases} t = & 1+2t & (1) \\ t^2 = & 1+6t & (2) \\ t^3 = & 1+14t & (3) \end{cases}$$ (1) $\Rightarrow t = -1$, which does not satisfy (2) or (3), so the particles do not collide.

   (b) Do their paths intersect?

   Solution: For the paths to intersect, we need to find a value for t and a value for s such that $\vec{r}_1(t) = \vec{r}_2(s) \iff \langle t, t^2, t^3\rangle = \langle 1+2s, 1+6s, 1+14s\rangle$. Equating components gives

   $$\begin{cases} t = & 1+2s & (4) \\ t^2 = & 1+6s & (5) \\ t^3 = & 1+14s & (6) \end{cases}$$

   Substituting (4) into (5) yields $(1+2s)^2 = 1+6s \Rightarrow 2s(2s-1) = 0 \Rightarrow s = 0$ or $\frac{1}{2}$.
   In (4): $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$.
   Check these s and t pairs in (6):
   - $s = 0, t = 1 : LHS : 1^3 = 1, RHS : 1 + 14 \cdot 0 = 1 \Rightarrow$ paths intersect at the point $(1, 1, 1)$.
   - $s = \frac{1}{2}, t = 2 : LHS : 2^3 = 8, RHS : 1 + 14 \cdot \frac{1}{2} = 1 + 7 = 8 \Rightarrow$ paths intersect at the point $(2, 4, 8)$.

   (c) For each particle, calculate the:

   i. velocity.
   Solution: $\vec{v}_1 = \vec{r}_1^{\,\prime} = \langle 1, 2t, 3t^2\rangle$, $\vec{v}_2 = \vec{r}_2^{\,\prime} = \langle 2, 6, 14\rangle$

   ii. acceleration.
   Solution: $\vec{a}_1 = \vec{v}_1^{\,\prime} = \vec{r}_1^{\,\prime\prime} = \langle 0, 2, 6t\rangle$, $\vec{a}_2 = \vec{v}_2^{\,\prime} = \vec{r}_2^{\,\prime\prime} = \langle 0, 0, 0\rangle$

   iii. velocity for $t = 1$.
   Solution: $\vec{v}_1(1) = \vec{r}_1^{\,\prime}(1) = \langle 1, 2, 3\rangle$, $\vec{v}_2(1) = \vec{r}_2^{\,\prime}(1) = \langle 2, 6, 14\rangle$

   iv. speed for $t = 1$.
   Solution: $|\vec{v}_1(1)| = |\vec{r}_1^{\,\prime}(1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$,
   $|\vec{v}_2(1)| = |\vec{r}_2^{\,\prime}(1)| = \sqrt{2^2+6^2+14^2} = \sqrt{236} = 2\sqrt{59}$

7. For the curve with parametric equations $x = t^2, y = 2t, z = \ln t$, for $t > 0$:

   (a) Find parametric equations for the tangent line to the curve at the point $(1, 2, 0)$.

   Solution: The vector equation for the curve is $\vec{r}(t) = \langle t^2, 2t, \ln t\rangle \Rightarrow \vec{r}^{\,\prime}(t) = \langle 2t, 2, 1/t\rangle$. The point $(1, 2, 0)$ corresponds to $t = 1$, so the tangent vector there is $\vec{r}^{\,\prime}(1) = \langle 2, 2, 1\rangle$. Thus, the tangent line is parallel to the vector $\langle 2, 2, 1\rangle$. The parametric equations are

   $$\begin{cases} x = 1+2t \\ y = 2+2t \\ z = 0+t = t \end{cases}$$

   (b) Find the length of the curve for $1 \le t \le 10$.

   Solution: From (a) we know that $\vec{r}^{\,\prime}(t) = \langle 2t, 2, 1/t\rangle$, then:

   $$|\vec{r}^{\,\prime}(t)| = \sqrt{(2t)^2+2^2+(1/t)^2} = \sqrt{4t^2+4+1/t^2} = \sqrt{\frac{4t^4+4t^2+1}{t^2}} = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{(2t^2+1)}{t} = 2t+\frac{1}{t},$$

   $$\Rightarrow L = \int_1^{10} |\vec{r}^{\,\prime}(t)| dt = \int_1^{10} 2t + \frac{1}{t}\, dt = t^2 + \ln t\Big]_1^{10} = 100 + \ln 10 - 1 - \ln 1 = 99 + \ln 10$$