Math 138 Physics Based Section Assignment 1

(Q1) Evaluate the following integrals:
i) $\int(\frac{1}{\sqrt{x}}\cos(\sqrt{x}))dx$
Use the change of variable method and let $u = \sqrt{x} = x^{\frac{1}{2}}$, then $du = \frac{1}{2\sqrt{x}}dx$ and using this in the integral we get

$$\int 2\cos(u)du$$

which can be immediately evaluated to give $2\sin(u) + C = 2\sin(\sqrt{x}) + C$.

ii) $\int_1^2(\frac{1}{\sqrt{x}}\cos(\sqrt{x}))dx$
We can do this one two ways. First we can just use the result of part i) with $C = 0$ to get

$$2\sin(\sqrt{2}) - 2\sin(1).$$

If we had to do it from scratch we woud use the same change of variables, but we would have to change the bounds on the integral because $x = 1$ means $u = 1$ while $x = 2$ means $u = \sqrt{2}$.

iii)$\int \frac{t}{1+t^2}dt$
Again by the change of variable method let $u = 1 + t^2$ then $du = 2tdt$ and so the integral becomes

$$\frac{1}{2}\int \frac{1}{u}du$$

and we evaluate to get $(1/2)\ln(u) + C = (1/2)\ln(1 + t^2) + C$.

iv) $\int \frac{t}{1+t^4}dt$
Again by change of variables, but this time we want to make something that looks like the derivative of arctan. To do this let $u = t^2$, then $du = 2tdt$ and the integral becomes

$$\int \frac{t}{1+u^2}\frac{du}{2t} = \frac{1}{2}\int \frac{1}{1+u^2}du$$

and this gives $(1/2)\arctan(u) + C = (1/2)\arctan(t^2) + C$.

v) $\int \frac{7t-3}{t(t-1)}dt$. For what values of t is the answer valid
Here we use partial fractions:

$$\frac{7t-3}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

and this gives

$$\frac{(A+B)t - A}{t(t-1)}$$

so that we have

$$(A+B)t - A = 7t - 3.$$

Equating the coefficients gives $A = 3$ and $B = 4$. Thus the integral is

$$\int \left(\frac{3}{t} + \frac{4}{t-1} \right) dt$$

which can be integrated to give $3\ln(t) + 4\ln(t-1) + C = \ln t^3(t-1)^4 + C$. Since the logarithm can only take in positive inputs we must have $t > 1$.

vi) $\int \frac{1-3t}{t^2} dt$
This one just requires some algebra

$$\int \left(\frac{1}{t^2} - \frac{3}{t} \right) dt$$

which can be integrated to give $-(1/t) - 3\ln(t) + C$.

vii) $\int \frac{5t^2+2}{t^4+2t^2} dt$
This is partial fractions again but with a twist, so write

$$\frac{5t^2+2}{t^4+2t^2} = \frac{A}{t^2} + \frac{B}{t^2+2}$$

so that

$$(A+B)t^2 + 2A = 5t^2 + 2$$

and upon equating coefficients we get $A = 1$ and $B = 4$ so that the integral becomes

$$\int \frac{1}{t^2} dt + \int \frac{4}{t^2+2} dt$$

where I write two integrals to make the next step easier. The first integral is easy and gives $-(1/t) + C$ but the second needs another change of variables. First factor out a 2 to get

$$\frac{4}{2} \int \frac{1}{(\frac{t}{\sqrt{2}})^2 + 1} dt$$

and now let $u = t/\sqrt{2}$ so that $dt = \sqrt{2}du$ and the integral becomes

$$2 \int \frac{\sqrt{2}}{u^2+1} du$$

and this can be solved to get $(2\sqrt{2})\arctan(u) = (2\sqrt{2})\arctan(t/\sqrt{2})$ and the final answer is thus

$$-(1/t) + 2\sqrt{2}\arctan\left(\frac{t}{\sqrt{2}}\right) + C$$

viii) $\int t \sin t dt$
This is a standard integration by parts question. We want to integrate the sine and differentiate the t:

$$\int t \sin t dt = -t\cos(t) + \int \cos(t) dt = -t\cos(t) + \sin(t) + C$$

ix) $\int_2^4 m \sin(m) dm$
Again there are two ways to go. One could evaluate as you go or just use the antiderivative. I prefer the latter, and get

$$\int_2^4 m\sin(m)dm = (-m\cos(m) + \sin(m))|_{m=2}^{m=4}$$

which gives

$$-4\cos(4) + \sin(4)) + 2\cos(2) - \sin(2))$$

x) $\int_0^5 \exp(-t)\sin(t)dt$
This one is a tough one, but everyone has to do it at least once. You might recall that the function inside the integral is an oscillation (the sine) and an envelope (the decaying exponential). The envelope gives the behaviour of the oscillations (decaying in this case). Again I do the antiderivative first. I choose to integrate the sine and differentiate the exponential:

$$\int \exp(-t)\sin(t)dt = -\cos(t)\exp(-t) - \int \cos(t)\exp(-t)dt$$

and now I do integration by parts again, making sure I integrate the cosine:

$$\int \exp(-t)\sin(t)dt = -\cos(t)\exp(-t) - \left[\sin(t)\exp(-t) + \int \sin(t)\exp(-t)dt\right]$$

and simplifying we get

$$\int \exp(-t)\sin(t)dt = \frac{1}{2}\left[-\cos(t)\exp(-t) - \sin(t)\exp(-t)\right].$$

Thus

$$\int_0^5 \exp(-t)\sin(t)dt = \frac{1}{2}\left[-\cos(5)\exp(-5) - \sin(5)\exp(-5) + \cos(0)\exp(0) + \sin(0)\exp(0)\right]$$

and finally

$$\frac{1}{2}\left[-\cos(5)\exp(-5) - \sin(5)\exp(-5) + 1\right]$$

(Q2)i) Find the area beneath the graph of $g(x) = x^2$ and above the horizontal line $y = -1$ for $0 \le x \le 5$

This is just the integral of $f(x) = g(x) - (-1)$ from $x = 0$ to $x = 5$:

$$\int_0^5 (x^2 + 1)dx = (\frac{x^3}{3} + 1)|_0^5 = \frac{140}{3}.$$

ii) Find the area to the left of the curve $g(x) = x^2$ and to the right of the y-axis for $0 \leq x \leq 2$. Here we just want to rewrite things in terms of y. This means inverting $g(x)$. For this case this is easy (definitely not the typical case). $h(y) = \sqrt{y}$ and $x = 0$ means $y = 0$ and $x = 2$ means $y = 4$. The area in question is thus

$$\int_0^4 h(y)dy = \frac{2}{3}y^{3/2}|_0^4 = \frac{16}{3}$$

iii) Find the area below the graph of $f(x) = \sin(\pi x)$ and above the x-axis.
Since sine is below zero just as much as above zero, and is a periodic function the integral over any number of entire periods is zero.

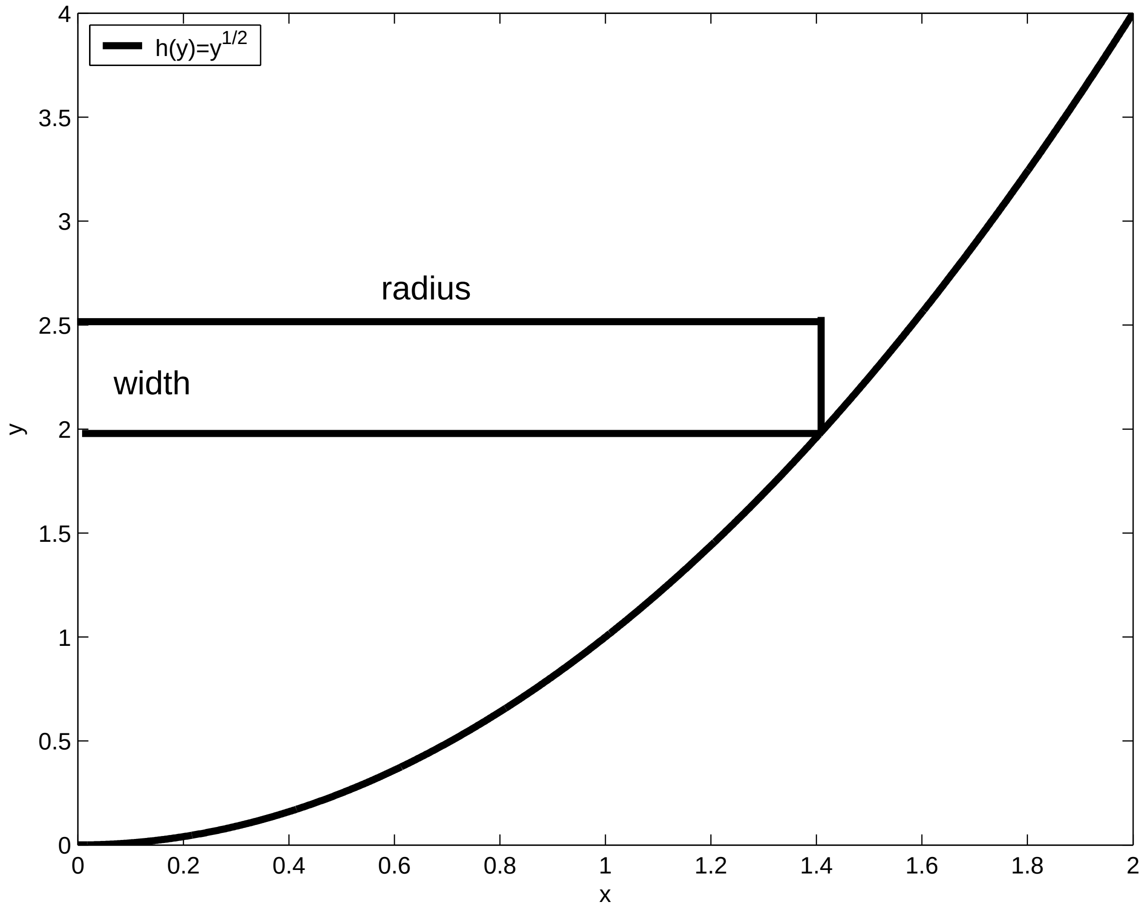
iv) Use your answer in part iii) to find the volume of the same region rotated about the y-axis. Include a clear sketch of the Riemann sums.
There are several choices here. My favourite is to think of this as a bunch of disks stacked one up on top of the next. Each has a radius of $h(y)$ and a thickness of Δy and thus a volume of

$$\pi h(y)^2 \Delta y$$

and the total volume will be

$$\int_0^4 \pi y dy = \frac{\pi}{2}y^2|_0^4 = 8\pi$$

Here is my Riemann Sum picture.



(Q3)i) Define

$$f(x) = \int_1^x \exp(-t)dt$$

Find $f'(x)$. Did you need to do any integration?
This is just a FTC II question. The FTC tells us

$$\int_1^x \exp(-t)dt = g(x) - g(1)$$

for some function $g(.)$ which we are usually asked to find. Now what do you actually know about $g(.)$? Well $g'(M) = \exp(-M)$, so in this case we could do the integral. But we don't have to because

$$f'(x) = g'(x) - \frac{dg(1)}{dx} = \exp(-x).$$

ii) Repeat i) for

$$f(x) = \int_1^x \exp(-t^2)dt$$

Hint: You can't do the integral.
By the argument from part i) we get

$$f'(x) = \exp(-x^2)$$

iii) If

$$f(x) = \int_1^x g(t)dt$$

find $f'(x)$ in terms of $g(.)$.

Here I am messing with the notation again. If you go through the argument from part i) again you will need a different label for the antiderivative because $g(.)$ is now taken. That's OK because notation is just a tool and here we get

$$f'(x) = g(x)$$

iv) Repeat the above for

$$f(x) = \int_1^{x^2} g(t)dt$$

find $f'(x)$ in terms of $g(.)$.
Now I really do need to go through the argument, so let's say that the FTC gives us

$$f(x) = G(x^2) - G(1)$$

where $G'(m) = g(m)$. Then I use the Chain Rule to get

$$f'(x) = G'(x^2)\frac{dx^2}{dx} - \frac{dG(1)}{dx} = 2xG'(x^2)$$

v) Now let's say we have an x inside the integral. Say

$$f(x) = \int_1^2 (x^2 t + \sin(xt))dt$$

find $f'(x)$ as an integral (DO NOT evaluate the integral).

Here the bounds of integration don't involve x so that I just bring the derivative inside the integral:

$$f'(x) = \int_1^2 \frac{d}{dx}(x^2 t + \sin(xt))dt$$

or by power rule and chain rule

$$f'(x) = \int_1^2 2xt + t\cos(xt))dt.$$

(Q4) Despite its apparent simplicity and usefulness (in statistics for example) the improper integral

$$\int_0^\infty \exp(-x^2)dx$$

is not possible to carry out using elementary functions (something worth thinking about, actually). Nevertheless we could at least try to get some estimates:
i) Using five boxes which split the interval $[0, 4]$ into equal width boxes, approximate the above integral. By making an appropriate choice can you make sure that your estimate is a lower bound?

By choosing the right end-point of each sub-interval you are guaranteed a lower bound because the function $f(x) = \exp(-x^2)$ is decreasing. Of course by choosing $0 \le x \le 4$ we are ignoring any contribution to the area from $x > 4$. The estimate will thus be

$$A \approx 0.8[f(0.8) + f(1.6) + f(2.4) + f(3.2) + f(4)]$$

ii) Since we are ignoring the entire set of inputs $x > 4$ we cannot make our approximation an upper bound with our existing definition of integral. You might guess that by allowing yourself to treat the region $x > 4$ as a single box you could find an upper bound for the Riemann sum. Show that this approach does not work.

Here the key thing to notice is that the region $x > 4$ would be a box of infinite length, and this is not good news as far as getting a finite area estimate.

(Q5)i) Show that the function

$$g(x) = \int_{-10}^{x} |t| dt$$

has a valid derivative for all $x > -10$. Explain why this is possible with reference to the fact that $|x|$ does not have a derivative at $x = 0$.

We can use our results above to find

$$g'(x) = |x|$$

and thus $g'(x)$ is a continuous function. Note that $g''(x)$ exists everywhere EXCEPT $x = 0$. In general the integral is a "smoothing" operation why the derivative can introduce discontinuities, as in the case of $|x|$.

ii) Find $\int |x|dx$ for $x \ge 0$ and for $x < 0$.

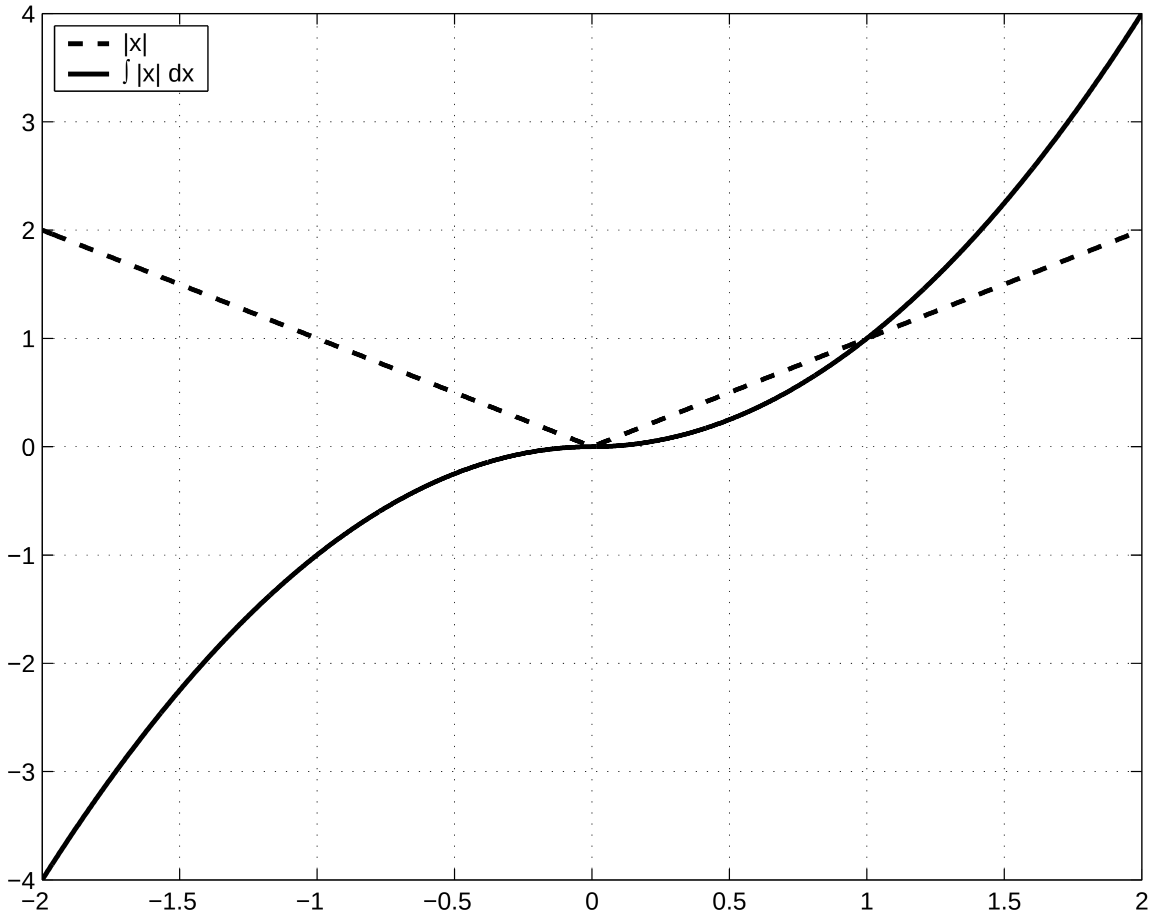
For $x > 0$ get

$$\int |x|dx = \int x dx = \frac{1}{2}x^2 + C$$

while for $x < 0$ get

$$\int |x|dx = \int (-x)dx = -\frac{1}{2}x^2 + C$$

iii) Set the arbitrary constants in part ii) to zero and sketch the resulting function on $-2 \leq x \leq 2$. On your sketch also include $|x|$.



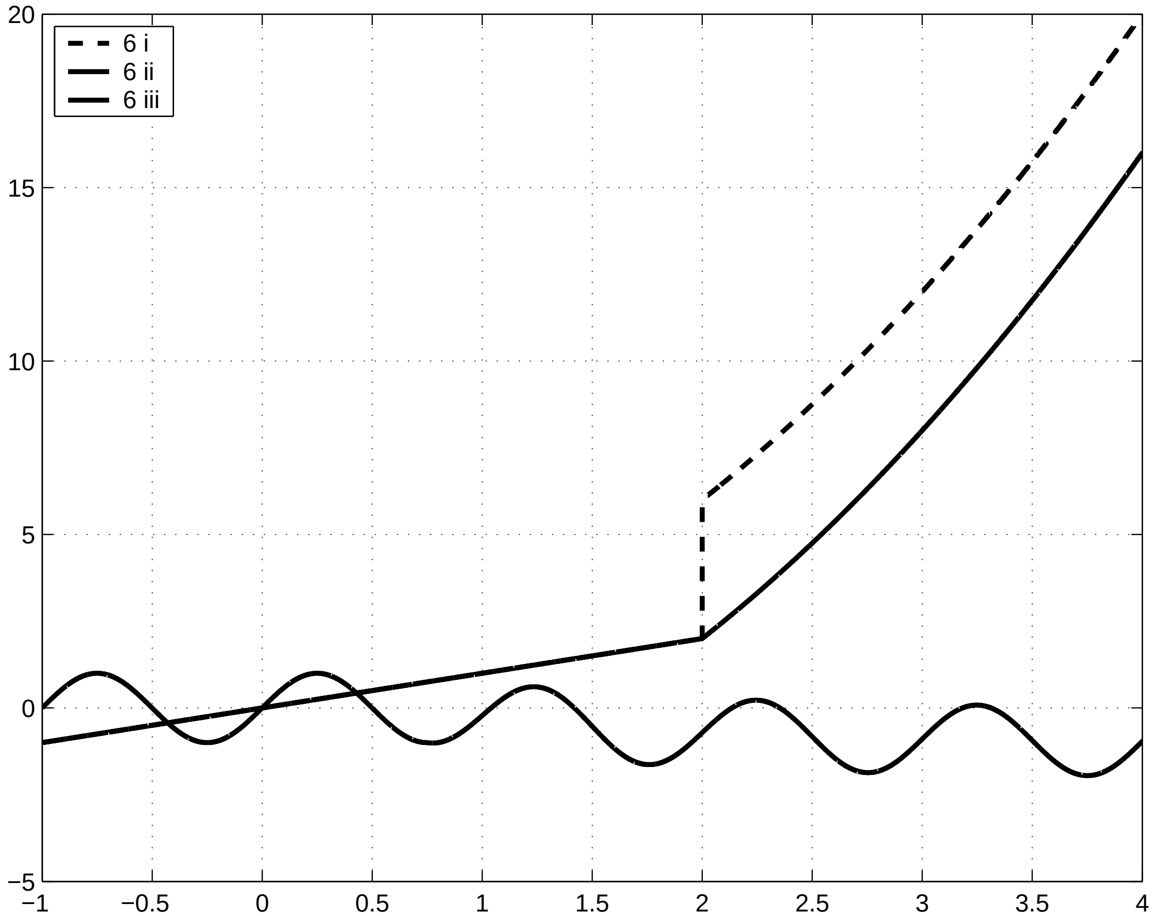
(Q6) Consider the following functions with potential jump discontinuities.

$$f(t) = t + t^2 H(t-2),$$

$$g(t) = t + (t^2 - 2)H(t-2),$$

$$p(t) = \sin(2\pi t) + H(t - 0.75)\exp(-t + 0.75).$$

i) Sketch all three functions and identify any points of discontinuity.



Only $f(t)$ has a jump, and it occurs at $t = 2$.

ii) Find

$$\int_0^4 f(t)dt$$

Here we split the integral into two parts:

$$\int_0^4 f(t)dt = \int_0^2 tdt + \int_2^4 (t + t^2)dt = 2 + \frac{74}{3} = \frac{80}{3}$$

(Q7) Recall that integrals whose domain went out to infinity required that the function being integrated goes to zero fast enough (for example $f(x) = 1/x$ does not go to zero fast enough). We now consider functions which "blow up" at some point. To be concrete we consider $g(x) = x^{-\alpha}$ for $\alpha > 0$ near the point $x = 0$. The integral in question will be

$$\int_0^1 g(x)dx$$

i) Define the above improper integral using a limit (your notes should be helpful).

$$\int_0^1 g(x)dx = \lim_{\epsilon \longrightarrow 0} \int_\epsilon^1 g(x)dx$$

provided that the limit exists.

ii) Show what happens when $\alpha = 1/2$, 1 and 2.

When $\alpha = 1/2$ get

$$\int_\epsilon^1 g(x)dx = -2\sqrt{\epsilon} + 2$$

and as $\epsilon \longrightarrow 0$ this limit does exist and gives.

$$\int_0^1 g(x)dx = 2$$

When $\alpha = 1$ get

$$\int_\epsilon^1 g(x)dx = \ln(1) - \ln(\epsilon) = \ln(1/\epsilon)$$

but as $\epsilon \longrightarrow 0$ this limit does not exist.

When $\alpha = 1$ get

$$\int_\epsilon^1 g(x)dx = -1 + \frac{1}{\epsilon}$$

but as $\epsilon \longrightarrow 0$ this limit does not exist.

iii) Based on ii) make a conjecture on the general result, and provide an explanation why you guessed what you did.
It looks like $\alpha < 1$ will give valid integrals because $g(x)$ goes to infinity "slowly enough".

iv) By defining a new variable $y(x)$ show that you can convert the above case of improper integrals to the case discussed in your notes.
Let $y = 1/x$ and this gives us an integral from $y = 1$ to $y = \infty$, or

$$\int_0^1 g(x)dx = \int_1^\infty G(y)dy$$

v) Based on your above work define the two-side improper integral

$$\int_{-1}^1 g(x)dx$$

Here we split things up into two as

$$\int_{-1}^1 g(x)dx = \int_{-1}^0 g(x)dx + \int_0^1 g(x)dx$$

Now we can define each of the pieces with a limit:

$$\int_{-1}^{0} g(x)dx = \lim_{b \longrightarrow 0} \int_{-1}^{b} g(x)dx$$

$$\int_{0}^{1} g(x)dx = \lim_{a \longrightarrow 0} \int_{a}^{1} g(x)dx$$

We must be careful to use two limits because it is possible that one integral could converge, but not the other. The two sided improper integral only exists if both limits exist.