MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 5 (Applications of DEs)

NOT TO BE HANDED IN

General

1. Let $Q(t) = Q_0e^{kt}$ represent the quantity of a substance (*e.g.*, monetary investment, radioactive material, bacteria in a colony, *etc.*) at time t.
   (a) What does Q_0 represent?
   (b) Assume $k > 0$. What is the doubling time for this quantity?
   (c) Assume $k < 0$. What is the half-life of this quantity?

Administering Medicine

2. A drug is infused into a patient's bloodstream at a constant rate r (the *infusion rate*) and is eliminated from the bloodstream at a rate proportional to the amount of drug present at time t. Initially the patient's blood contains no drug.
   (a) Set up and solve a differential equation (DE) for the amount of drug $A(t)$ present in the patient's bloodstream at time t.
   (b) Calculate $\lim_{t\to\infty} A(t)$. (This is called the steady-state level of medication.)
   (c) Suppose the infusion rate is set at $2\,mg/hr$ and a blood sample is taken indicating there are $1.5\,mg$ of the drug present after 1 hour and $2.5\,mg$ present after two hours.
      i. How much of the drug remains after six hours? (Note that $\ln 1.5 \approx 0.405$ and $1.5^{-6} \approx 0.088$.)
      ii. What is the steady-state level of medication?
      iii. Suppose the desired steady-state level of medication in the patient's bloodstream is $8\,mg$. How should the infusion rate be adjusted to achieve this goal?

Decomposition

3. Suppose the half-life of a certain type of plastic bag is 200 years. Let $B(t)$ be the amount (in kg) of these plastic bags in a number of landfills at time t in years.
   (a) If $B(2) = 10^6\,kg$, then when is $B(t) = 1000\,kg$?
   (b) When will these plastic bags decay to 10% of the original amount?

   (Note that $\ln 10 \approx 2.303$, $\ln 2 \approx 0.693$.)

Investments

4. (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
   (b) What is the equivalent annual interest rate?

   (Note that $\ln 2 \approx 0.693$, $e^{0.06} \approx 1.0618$.)

Dilution Models / Mixing Problems

5. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

6. A 300-ℓ tank initially contains a concentration of 10 kg of salt dissolved in 200 ℓ of water. Suppose that 12 ℓ of brine containing 1 kg of dissolved salt per ℓ enters the tank every minute and that the mixture (kept uniform by stirring) exits the tank at the rate of 8 ℓ per minute.
   (a) Find the rate of change of salt in the tank at time t.
   (b) What is the concentration of salt in the tank after 10 minutes?
   (c) How much salt will be in the tank at the instant the tank begins to overflow?

Chemical Reactions

7. Assume that single molecules of two *reactants* (or chemicals), $\mathcal{A}$ and $\mathcal{B}$, react to form a molecule of the another chemical, C, called the *product*: $\mathcal{A} + \mathcal{B} \to C$.

   Law of mass action: If the temperature is kept constant, the rate of reaction is proportional to the product of the (instantaneous) concentrations of the substances which are reacting.

   Let $[\mathcal{A}]$, $[\mathcal{B}]$, and $[C]$ denote the concentrations of $\mathcal{A}$, $\mathcal{B}$, and C, respectively, at time t hours. The concentrations are expressed in moles per ℓ (where 1 mole = 6.022×10^{23} molecules).
   (a) State an equation representing the rate of production of $[C]$ in terms of $[\mathcal{A}]$ and $[\mathcal{B}]$.
   (b) Assume that the formation of C requires twice as much of $\mathcal{A}$ as that of $\mathcal{B}$. If 150 moles/ℓ of $\mathcal{A}$ and 300 moles/ℓ of $\mathcal{B}$ are present initially, and if 100 moles/ℓ of C are formed in 20 minutes, find the concentration of C at any time.
   (c) Find the limiting value of the concentration C. (That is, if the reaction continues indefinitely, what is the final concentration of C?)

Temperature Variation

8. A sphere with radius $1m$ has temperature $15°C$. It lies inside a concentric sphere with radius $2m$ and temperature $25°C$. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the second-order DE

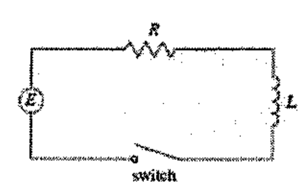
$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0.$$

   If we let $S = dT/dr$, then S satisfies a first-order DE. Solve the new DE to find an expression for the temperature $T(r)$ between the two spheres.

9. A cup of hot chocolate is taken outdoors where the temperature is $-5°C$. After 10 minutes, its temperature is $70°C$, and 10 minutes after that, its temperature is $50°C$. What was the original temperature of the drink?

Electric Circuits

10. In the circuit shown, a generator supplies a voltage of $E(t) = 40\sin 60t$ volts (V), the inductance L is 1 henry (H), the resistance R is 20 ohms (Ω), and the initial current is $I(0) = 1$ ampere (A). The current $I(t)$ satisfies the DE $LI'(t) + RI(t) = E(t)$. Find $I(t)$.



MA128
Fall '06

Solutions: Asst. 5 (not due)

1. $Q = Q_0 e^{kt}$

a) $Q_0 =$ initial value of Q, i.e., at $t=0$.

b) $k>0$: $Q = Q_0 e^{kt}$: when is $Q = 2Q_0$?

$$2Q_0 = Q_0 e^{kt_d}$$
$$2 = e^{kt_d}, \quad Q_0 \neq 0.$$
$$\ln 2 = kt_d$$
$$\therefore \boxed{t_d = \frac{\ln 2}{k}}$$

c) $k<0$: $Q = Q_0 e^{kt}$: when is $Q = \frac{1}{2} Q_0$?

$$\frac{1}{2}Q_0 = Q_0 e^{kt_h}$$
$$\frac{1}{2} = e^{kt_h}$$
$$\ln\left(\frac{1}{2}\right) = kt_h$$
$$\boxed{t_h = \frac{\ln\left(\frac{1}{2}\right)}{k}} = \frac{\cancel{\ln 1}^{\,0} - \ln 2}{k} = \frac{-\ln 2}{k}$$

2. a) $\boxed{\frac{dA}{dt} = r - kA}$, $A(0)=0$ (IVP)

DE: separable & linear.

separable

$$\frac{dA}{r-kA} = dt$$
$$-\frac{1}{k}\ln|r-kA| = t + c_1$$
$$\ln|r-kA| = -kt - kc_1$$

$$|r-kA| = e^{-kt-kc_1} = c_2 e^{-kt}, \quad (c_2 = e^{-kc_1})$$

$$r-kA = \pm c_2 e^{-kt} = c_3 e^{-kt}, \quad (\pm c_2 = c_3)$$

$$-kA = -r + c_3 e^{-kt}$$

$$A = \frac{r}{k} - \frac{c_3}{k} e^{-kt} = \frac{r}{k} + c_4 e^{-kt}, \quad \left(-\frac{c_3}{k} = c_4\right)$$

$$\Rightarrow \boxed{A = \frac{r}{k} + c_4 e^{-kt}}$$

$$A(0) = 0 = \frac{r}{k} + c_4 e^0 = \frac{r}{k} + c_4 \Rightarrow c_4 = -\frac{r}{k}$$

$$\Rightarrow \boxed{A = \frac{r}{k}\left(1 - e^{-kt}\right)}$$

OR, linear: $\frac{dA}{dt} + kA = r \rightarrow P(t) = k, \; Q(t) = r$

$$\Rightarrow I(t) = e^{\int P(t)\,dt} = e^{\int k\,dt} = e^{kt}$$

$$\Rightarrow \underbrace{e^{kt}\left(\frac{dA}{dt} + kA\right)} = e^{kt} r$$

$$\frac{d}{dt}\left(e^{kt} \cdot A\right) = e^{kt} r$$

$$e^{kt} A = r\int e^{kt}\,dt = \frac{r}{k} e^{kt} + c_5$$

$$\boxed{A = \frac{r}{k} + c_5 e^{-kt}}$$

$$A(0) = 0 = \frac{r}{k} + c_5 e^0 = \frac{r}{k} + c_5 \Rightarrow c_5 = -\frac{r}{k}$$

$$\therefore \boxed{A = \frac{r}{k}\left(1 - e^{-kt}\right)}$$

2b) $\lim_{t\to\infty} A = \lim_{t\to\infty} \frac{r}{k}(1-e^{-kt}) = \boxed{\frac{r}{k}}$

c) $r = 2$; $A(\cancel{1}) = 1.5$, $A(2) = 2.5$ $\longrightarrow$ solve for k:

i) $A = \frac{2}{k}(1-e^{-kt})$

$$\begin{cases} A(1) = 1.5 = \frac{2}{k}(1-e^{-k}) \Rightarrow \frac{2}{k} = \frac{1.5}{1-e^{-k}} \\ A(2) = 2.5 = \frac{2}{k}(1-e^{-2k}) \Rightarrow \frac{2}{k} = \frac{2.5}{1-e^{-2k}} \end{cases}$$

$\Rightarrow \frac{1.5}{1-e^{-k}} = \frac{2.5}{1-e^{-2k}} \Rightarrow \underbrace{\frac{2.5}{1.5}}_{5/3} = \frac{1-e^{-2k}}{1-e^{-k}} = \frac{(1+e^{-k})\cancel{(1-e^{-k})}}{\cancel{(1-e^{-k})}}$

$\Rightarrow \frac{5}{3} = 1+e^{-k}$

$\Rightarrow \frac{5}{3} - 1 = e^{-k}$

$\Rightarrow \frac{2}{3} = e^{-k} \Rightarrow \ln\frac{2}{3} = -k \Rightarrow \boxed{k = -\ln\frac{2}{3} = \ln\frac{3}{2}}$

(since $-\ln\frac{2}{3} = \ln\left(\frac{2}{3}\right)^{-1} = \ln\frac{3}{2}$)

$\Rightarrow \boxed{A = \frac{2}{\ln\frac{3}{2}}\left(1-e^{-(\ln 3/2)t}\right)} \overset{\text{OR}}{=} \frac{2}{\ln\frac{3}{2}}\left(1-\left(\frac{2}{3}\right)^{t}\right) \overset{\text{OR}}{=} \frac{2}{\ln\frac{3}{2}}\left(1-\left(\frac{3}{2}\right)^{-t}\right)$

$A(6) = \frac{2}{\ln\frac{3}{2}}\left(1-e^{-(\ln\frac{3}{2})6}\right) = \frac{2}{\ln\frac{3}{2}}\left(1-\left(\frac{3}{2}\right)^{-6}\right)$

$\approx \frac{2}{0.405}(1-0.088)$

$\approx 4.938(0.912)$

$\approx \underline{\underline{4.504}}$ [mg]

2 c) ii) From b) : $\frac{r}{k} = \frac{2}{\ln 1.5} \approx \frac{2}{0.405} \approx \underline{4.94\,mg}$

iii). From b) $\frac{r}{\ln 1.5} = 8 \Rightarrow r = 8\ln 1.5 \approx 8(0.405) \approx \underline{\underline{3.24\,mg}}$

(assumes k fixed at $\ln 1.5$)

3. $\frac{dB}{dt} = kB,\ k<0 \Rightarrow B = B_0 e^{kt},\ B_0 = B(0),\ k<0.$

given: half-life is 200 years

From 1c) : $t_h = -\frac{\ln 2}{k} = 200$

$\Rightarrow k = -\frac{\ln 2}{200}$

$\Rightarrow \boxed{B = B_0 e^{-\frac{\ln 2}{200}t}}$

ALSO: $\boxed{B = B_0 2^{-\frac{t}{200}}}$

a) $B(2) = 10^6 = B_0 e^{-\frac{\ln 2}{200}\cdot 2} = B_0 e^{-\frac{\ln 2}{100}}$

$\Rightarrow B_0 = \frac{10^6}{e^{-\frac{\ln 2}{100}}} = 10^6 e^{\frac{\ln 2}{100}}$ (or $10^6 2^{\frac{1}{100}}$)

$\Rightarrow \boxed{B = 10^6 e^{\frac{\ln 2}{100}} e^{-\frac{\ln 2}{200}t} = 10^6 e^{\frac{\ln 2}{100}(1-\frac{t}{2})}}$

$B = 1000 = 10^6 e^{\frac{\ln 2}{100}(1-\frac{t}{2})}$

$\Rightarrow \frac{10^3}{10^6} = e^{\frac{\ln 2}{100}} e^{\frac{-\ln 2}{200}t}$

$\Rightarrow \frac{10^{-3}}{e^{(\ln 2)/100}} = e^{\frac{-\ln 2}{200}t}$

$$\Rightarrow \ln\left(\frac{10^{-3}}{e^{(\ln 2)/100}}\right) = \ln e^{-\frac{\ln 2}{200}t} = -\frac{\ln 2}{200}t$$

$$\Rightarrow t = \frac{\ln\left(\frac{10^{-3}}{e^{(\ln 2)/100}}\right)}{-\frac{\ln 2}{200}} = -\frac{200}{\ln 2}\left(\ln 10^{-3} - \ln e^{\frac{\ln 2}{100}}\right)$$

$$= \frac{600}{\ln 2}\ln 10 + \underbrace{\frac{200}{\cancel{\ln 2}}\left(\frac{\cancel{\ln 2}}{100}\right)}_{=2}$$

$$\approx \frac{600}{0.693}(2.303) + 2$$

$$\approx \underline{1995.16} \text{ years.}$$

b) when is $B = 0.10 B_0$?

$$B = 0.1 B_0 = B_0 e^{(-(\ln 2)/200)t}$$

$$\Rightarrow 0.1 = e^{-((\ln 2)/100)t}$$

$$\ln 0.1 = \ln\left(e^{-\frac{\ln 2}{100}t}\right) = -\frac{\ln 2}{100}t$$

$$\Rightarrow t = -\frac{\ln 0.1}{\frac{\ln 2}{100}} = \frac{-100(\ln 0.1)}{\ln 2} = \frac{-100 \ln \frac{1}{10})}{\ln 2}$$

$$= \frac{-100(\ln 1 - \ln 10)}{\ln 2} = \frac{-100(-\ln 10)}{\ln 2} \quad (\ln 1 = 0)$$

$$\approx \frac{100 \cdot (2.303)}{0.693}$$

$$\approx \underline{\underline{664.4 \text{ years}}}$$

4. a) From 1b) $t_d = \frac{\ln 2}{0.06} \approx \frac{0.693}{0.06} = \boxed{11.55 \text{ years.}}$

OR

Since $A = A_0 e^{rt} = A_0 e^{0.06t}$, $A(t)$ is amount or value of investment after t years, A_0 is initial investment, $r = 0.06$ is the continuous compounded interest rate.

when is $A = 2A_0$?

$\Rightarrow 2A_0 = A_0 e^{0.06t}$

$2 = e^{0.06t}$

$\ln 2 = \ln e^{0.06t} = 0.06t$

$\Rightarrow t = \frac{\ln 2}{.06}$ etc. (as above)

b) $A = A_0 (1+R)^t$, where R is the annual interest rate

From a), $A = A_0 e^{rt} = A_0 e^{0.06t}$, equate:

$A_0 (1+R)^t = A_0 e^{0.06t}$

$\Rightarrow (1+R)^t = (e^{0.06})^t$

$\Rightarrow 1+R = e^{0.06}$ since $a^t = b^{ct} = (b^c)^t \Leftrightarrow a = b^c$

$\Rightarrow R = e^{0.06} - 1$

$\approx 1.0618 - 1$

$= 0.0618$ or $\boxed{6.18\%}$

5. $y(t)$ - amount of CO_2 initially

$\Rightarrow y(0) = 0.0015(180) = 0.27 m^3$

- air in room always $180 m^3$

$\Rightarrow$ percentage of CO_2 in room at time t is: $\frac{y}{180} \cdot 100$

- change in CO_2 in time:

$$\frac{dy}{dt} = (0.0005)\left(2\frac{m^3}{min}\right) - \frac{y}{180}\left(2\frac{m^3}{min}\right)$$

$$= 0.001 - \frac{y}{90}$$

or $\boxed{\frac{dy}{dt} = \frac{9-100y}{9000}}$ (m^3/min) — separable & linear

<u>Separable</u>:

$$\frac{dy}{9-100y} = \frac{dt}{9000}$$

$$-\frac{1}{100}\ln|9-100y| = \frac{t}{9000} + C$$

<u>IC:</u> $y(0) = 0.27$:

$$-\frac{1}{100}\ln|9-27| = \frac{0}{9000} + C$$

$$-\frac{1}{100}\ln 18 = C$$

$$\Rightarrow -\frac{1}{100}\ln|9-100y| = \frac{t}{9000} - \frac{1}{100}\ln 18$$

$$\Rightarrow \ln|9-100y| = -\frac{t}{90} + \ln 18 = \ln e^{-t/90} + \ln 18 \qquad \begin{cases} \ln e^a = a, \\ \ln e^{-t/90} \\ = -t/90 \end{cases}$$

$$\Rightarrow \ln|9-100y| = \ln\left(18e^{-t/90}\right)$$

$$\Rightarrow |9-100y| = 18e^{-t/90}$$

To simplify $|9-100y|$ as $+(9-100y)$ or $-(9-100y)$ we can consider two ideas:

(1) since $y(t)$ is continuous, $y(0)=0.27$, and $18e^{-t/90} \neq 0$, then $9-100y<0$ (else $y<\frac{9}{100}$ which contradicts $y(0)=.27$)

OR (2) Consider the equilibrium solution, $\frac{dy}{dt}=0$ for $\frac{9-100y^*}{9000}=0 \Rightarrow y^*=\frac{9}{100}=0.09$.

Solutions will approach this equilibrium solution from above or below it. Since $y(0)=0.27$ is larger than $y^*=0.09$, our (as yet unknown) solution will approach $y^*=0.09$ from above, which means that the slope of the Solution is negative (it must be a decreasing solution to approach y^* from above).

$\Rightarrow \frac{dy}{dt}<0$ for IC's >0.09 (e.g., $y(0)=0.27$)

$\Rightarrow \frac{dy}{dt}=\frac{9-100y}{9000}<0 \Rightarrow 9-100y<0$

$\Rightarrow |9-100y| = -(9-100y) = -9+100y = 18e^{-t/90}$

$\Rightarrow 100y = 9+18e^{-t/90}$

$\Rightarrow \boxed{y = \frac{9}{100}+\frac{18}{100}e^{-t/90} = 0.09+0.18e^{-t/90}}$

OR *linear*: $\frac{dy}{dt} + \frac{y}{90} = 0.001$, $P(t) = \frac{1}{90}$

$\Rightarrow I(t) = e^{\int P(t)dt} = e^{\int \frac{dt}{90}} = e^{t/90}$

$\Rightarrow \underbrace{e^{t/90}\left(y' + \frac{y}{90}\right)} = 0.001e^{t/90}$

$\frac{d}{dt}\left(e^{t/90}y\right) = 0.001e^{t/90}$

$e^{t/90}y = 0.001\int e^{t/90}\,dt = \frac{0.001}{90}e^{t/90} + C$

$e^{t/90}y = 0.09e^{t/90} + C$

$y = 0.09 + Ce^{-t/90}$

$y(0) = 0.27 = 0.09 + Ce^{0} = 0.09 + C \Rightarrow C = 0.27 - 0.09 = 0.18$

$\therefore \boxed{y = 0.09 + 0.18e^{-t/90}}$

- Percentage of CO_2 in room is $\frac{y}{180}\cdot 100 = \left(\frac{.09 + .18e^{-t/90}}{180}\right)100$

$= \boxed{0.05 + 0.1e^{-t/90}}$

- Long run: $\lim_{t\to 0}\left(0.05 + 0.1e^{-t/90}\right) = \boxed{0.05}$,

$\Rightarrow$ CO_2 approaches 5% as time goes on

6. a) Let $S(t)$ be amount of salt in tank at time t; S in kg, t in minutes. $S(0) = 10$ kg.

$\Rightarrow \frac{dS}{dt}$ is the rate of change of salt in the tank at time t:

$$\underbrace{\frac{dS}{dt}}_{\text{net rate of change of salt}} = \underbrace{\text{inflow}}_{\text{gain of salt}} - \underbrace{\text{outflow}}_{\text{loss of salt}}$$

inflow:

Salt is added at the rate $\frac{12\ell}{\text{min}} \cdot \frac{1\text{kg}}{\ell} = 12\frac{\text{kg}}{\text{min}}$.

outflow:

At time t, there is $S(t)$ kg of salt in the tank and

$200 + (12-8)t = 200 + 4t$ ℓ of liquid in the tank

$\Rightarrow$ concentration of salt in tank at time t is $\frac{S}{200+4t}\left(\frac{\text{kg}}{\ell}\right)$

and salt exits at the rate

$$\left(\frac{S}{200+4t}\frac{\text{kg}}{\ell}\right)\left(\frac{8\ell}{\text{min}}\right) = \frac{8S}{200+4t} = \frac{2S}{50+t} \quad (\text{kg}/\text{min})$$

$$\Rightarrow \boxed{\frac{dS}{dt} = 12 - \frac{2S}{50+t}}$$ (not separable but linear)

b)

$$\frac{dS}{dt} + \frac{2}{50+t}S = 12, \quad P(t) = \frac{2}{50+t}$$

$$\Rightarrow I(t) = e^{\int P(t)\,dt} = e^{\int \frac{2}{50+t}dt} = e^{2\ln(50+t)} = e^{\ln(50+t)^2}$$

$$= (50+t)^2$$

(since $e^{\ln a^2} = a^2$)

$$\Rightarrow \underbrace{(50+t)^2\left(\frac{dS}{dt} + \left(\frac{2}{50+t}\right)S\right)} = 12(50+t)^2$$

$$\frac{d}{dt}\left((50+t)^2 S\right) = 12(50+t)^2$$

$$(50+t)^2 S = 12\int (50+t)^2\, dt$$

$$(50+t)^2 S = 12\,\frac{(50+t)^3}{3} + C$$

$$\Rightarrow S = \frac{\cancel{12}^{\,4}}{\cancel{3}}(50+t) + C(50+t)^{-2}$$

or $\boxed{S = 4(50+t) + \frac{C}{(50+t)^2}}$

$$S(0) = 10 = 4(50) + \frac{C}{(50)^2} = 200 + \frac{C}{2500}$$

$$\Rightarrow -190(2500) = C \Rightarrow C = -475000$$

$$\Rightarrow \boxed{S = 4(50+t) - \frac{475000}{(50+t)^2}}$$ ← salt (in kg) in tank at time t

• Concentration of salt at time t is $\frac{S(t)}{200+4t}$ $\left(\text{kg}/\ell\right)$

$$\Rightarrow \frac{S(10)}{200+4(10)} = \frac{4(50+10) - \frac{475000}{(60)^2}}{240} = \frac{389}{864} \approx 0.45\ \text{kg}/\ell$$

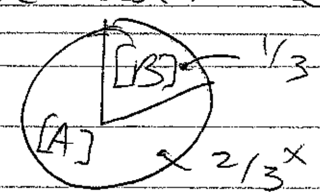
c) At t minutes, the tank contains $200+4t$ ℓ of liquid and begins to overflow when the liquid reaches the capacity of the tank, or 300ℓ: $200+4t = 300$

$\Rightarrow t = 25$ minutes

$$\Rightarrow S(25) = \frac{1940}{9} \approx 215.5\ \text{kg}$$

7. a) $\dfrac{d[C]}{dt} = k[A][B]$

b) To form x molecules of C we need twice as much of [A] as [B]:



If x represents moles/ℓ of C,

$\Rightarrow \boxed{150 - \frac{2}{3}x}$ is amount of A present when x moles/ℓ of C is formed.

Likewise, $\boxed{300 - \frac{x}{3}}$ is amount of B present.

$$\Rightarrow \frac{dx}{dt} = k\left(150 - \frac{2x}{3}\right)\left(300 - \frac{x}{3}\right) = \frac{k}{9}(450-2x)(900-x)$$

a separable DE:

$$\frac{dx}{(450-2x)(900-x)} = \frac{k}{9}\,dt$$

$$\int \frac{dx}{(450-2x)(900-x)} = \frac{k}{9}\int dt = \frac{k}{9}t + C_1$$

partial fractions:

$$\int \frac{dx}{(450-2x)(900-x)} =$$

$$= \frac{1}{675}\ln(450-2x)\left(-\frac{1}{2}\right) - \frac{1}{1350}\ln(900-x)(-1) + C_2$$

$$= \frac{1}{1350}\ln\left(\frac{900-x}{450-2x}\right) + C_2$$

{ don't need abs. values, since [A] + [B] > 0.

$$\Rightarrow \frac{1}{1350}\ln\left(\frac{900-x}{450-2x}\right) = \frac{k}{9}t + C_3 \qquad (C_3 = C_1 - C_2)$$

$$\Rightarrow \ln\left(\frac{900-x}{450-2x}\right) = \frac{1350}{9}kt + 1350c_3 = 150kt + c_4,$$

$$\frac{900-x}{450-2x} = e^{150kt+c_4} = c_5 e^{150kt}, \quad (1350c_3 = c_4) \quad (e^{c_4} = c_5)$$

$$\frac{900-x}{225-x} = 2c_5 e^{150kt}$$

IC $x(0)=0$: $\quad \frac{900}{225} = 4 = 2c_5 e^0 = 2c_5 \Rightarrow c_5 = 2$

$$\Rightarrow \frac{900-x}{225-x} = 4e^{150kt}$$

IC $x(1/3)=100$: (20 minutes = 1/3 hour)

$$\frac{900-100}{225-100} = \frac{800}{125} = 4e^{150k/3} = 4e^{50k}$$

$$\Rightarrow \boxed{\frac{8}{5} = e^{50k}}$$ — don't need to solve for k, we can use this result as is; since

$$\Rightarrow \frac{900-x}{225-x} = 4e^{150kt} = 4(e^{50k})^{3t} = 4\left(\frac{8}{5}\right)^{3t}$$

Solve ~~for x~~:

$$900-x = (225-x)4\left(\frac{8}{5}\right)^{3t} = 900\left(\frac{8}{5}\right)^{3t} - 4x\left(\frac{8}{5}\right)^{3t}$$

$$\Rightarrow \boxed{x = \frac{900\left(\frac{8}{5}\right)^{3t} - 900}{-1 + 4\left(\frac{8}{5}\right)^{3t}}}$$

c) $$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{900 - 900\left(\frac{8}{5}\right)^{-3t}}{-\left(\frac{8}{5}\right)^{-3t} + 4} = \frac{900}{4} = \underline{\underline{225}} \left(\frac{\text{moles}}{\ell}\right)$$

(multiply by $\frac{(8/5)^{-3t}}{(8/5)^{-3t}}$)

f. $S = \frac{dT}{dr} \Rightarrow \frac{dS}{dr} = \frac{d}{dr}\left(\frac{dT}{dr}\right) = \frac{d^2T}{dr^2}$

$$\Rightarrow \frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = \underbrace{\frac{dS}{dr} + \frac{2}{r}S = 0}_{\text{separable / linear DE}}$$

OR

Separable:

$\frac{dS}{dr} = -\frac{2}{r}S$

$\Rightarrow \frac{dS}{S} = -\frac{2}{r}dr$

$\int \frac{dS}{S} = \int -\frac{2}{r}dr$

$\ln|S| = -2\ln r + C$

(if $\frac{dT}{dr} > 0 \Rightarrow S > 0$)

$\Rightarrow S = e^{-2\ln r + C} = e^{\ln r^{-2}} e^{C}$

$\Rightarrow S = c_1 \frac{1}{r^2}$ $\quad (e^C = c_1)$

$\Rightarrow \frac{dT}{dr} = \frac{c_1}{r^2}$ (separable)

$dT = c_1 \int r^{-2} dr$

$T = -c_1 r^{-1} + c_2$

Linear DE:

$P(r) = \frac{2}{r}$

$I(r) = e^{\int P(r)dr}$

$= e^{\int \frac{2}{r}dr}$

$= e^{2\ln r}$, $r > 0$

$= e^{\ln r^2}$

$\therefore I = r^2$

$\Rightarrow r^2 S' + r^2 \frac{2}{r} S = 0 \cdot r^2 = 0$

$\Rightarrow \underbrace{r^2 S' + 2rS}_{} = 0$

$\frac{d}{dr}(r^2 S) = 0$

$\int \frac{d}{dr}(r^2 S)dr = \int 0 dr$

$r^2 S = c_1$

$S = c_1 r^{-2}$

(separable): $\frac{dT}{dr} = c_1 r^{-2}$

$\int dT = c_1 \int r^{-2} dr$

$T = -c_1 r^{-1} + c_2$

Both cases lead to applying ICs:

$$\begin{cases} T(1) = \boxed{15 = -c_1 + c_2} \\ T(2) = \boxed{25 = -\frac{c_1}{2} + c_2} \end{cases}$$

$\rightarrow$ subtracting: $-10 = -\frac{c_1}{2} \Rightarrow c_1 = 20$

$\Rightarrow 15 = -20 + c_2 \Rightarrow c_2 = 35$

$\therefore \boxed{T = -\frac{20}{r} + 35}$

9. Let $T(t)$ be temperature of hot chocolate, in °C, at time t, in minutes. Then, by Newton's law:

$$\frac{dT}{dt} = k(T - T_a),$$

where $T_a = -5°C$, is the ambient temperature.

$$\Rightarrow \boxed{\frac{dT}{dt} = k(T+5)}.$$ (separable & linear)

Also given: $T(10) = 70$, $T(20) = 50$

separable ← OR → linear

separable

$$\int \frac{dT}{T+5} = \int k\,dt$$

$$\ln|T+5| = kt + c$$

(if $T = 5$, $\frac{dT}{dt} = 0$, so $T^* = 5$ is an equilibrium solution)

$$T + 5 = e^{kt+c} = e^c e^{kt} = c_1 e^{kt} \quad (e^c = c_1)$$

$$\therefore \boxed{T = -5 + c_1 e^{kt}}$$

linear

$$\frac{dT}{dt} - kT = 5k, \quad P(t) = -k$$

$$\Rightarrow I(t) = e^{\int P(t)dt} = e^{\int -k\,dt} = e^{-kt}$$

$$\Rightarrow \underbrace{e^{-kt}(T' - kT)} = 5ke^{-kt}$$

$$\frac{d}{dt}(e^{-kt}T) = 5ke^{-kt}$$

$$e^{-kt}T = 5k\int e^{-kt}dt$$

$$e^{-kt}T = -5e^{-kt} + c$$

$$\boxed{T = -5 + c_1 e^{kt}}$$

Apply ICs:

$$\begin{cases} T(10) = 70 = -5 + c_1 e^{10k} \Rightarrow \frac{75}{c_1} = e^{10k} \\ T(20) = 50 = -5 + c_1 e^{20k} \Rightarrow \frac{55}{c_1} = e^{20k} = (e^{10k})^2 = \left(\frac{75}{c_1}\right)^2 \end{cases}$$

$$\Rightarrow c_1 = \frac{(75)^2}{55}$$

$$\Rightarrow \frac{75}{\frac{(75)^2}{55}} = e^{10k} \quad \Rightarrow \frac{55}{75} = e^{10k}$$

$$\Rightarrow \ln\left(\frac{55}{75}\right) = 10k$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{55}{75}\right)$$

$$\Rightarrow T = -5 + \frac{(75)^2}{55} e^{\ln\left(\frac{55}{75}\right)\frac{t}{10}}$$

IC

$$T(0) = -5 + \frac{(75)^2}{55} e^{\ln\left(\frac{55}{75}\right)\cdot\frac{0}{10}} = -5 + \frac{(75)^2}{55}\cdot 1$$

$$= \frac{1070}{11} \approx \boxed{97.27°C}$$

10. $I' + 20I = 40\sin 60t$ $\rightarrow$ linear DE: integ. factor is $e^{\int 20dt} = e^{20t}$

$$\rightarrow \underbrace{e^{20t}(I' + 20I)} = e^{20t} 40\sin 60t$$

$$\frac{d}{dt}\left(e^{20t} I\right) = 40e^{20t}\sin 60t$$

$$\Rightarrow \boxed{e^{20t} I = 40 \underbrace{\int e^{20t}\sin 60t\,dt}_{\text{let this be } R.}}$$

$R = \int e^{20t}\sin 60t\,dt$: IBP $\quad u = \sin 60t \quad v = \frac{e^{20t}}{20}$

$\quad du = 60\cos 60t \quad dv = e^{20t}dt$

$$\Rightarrow R = \frac{1}{20}e^{20t}\sin 60t - \frac{60}{20}\underbrace{\int e^{20t}\cos 60t\,dt}$$

IBP: $u = \cos 60t \quad v = \frac{e^{20t}}{20}$

$du = -60\sin 60t\,dt \quad dv = e^{20t}dt$

$$\Rightarrow R = \frac{1}{20}e^{20t}\sin 60t - 3\left[\frac{1}{20}e^{20t}\cos 60t + \frac{60}{20}\underbrace{\int e^{20t}\sin 60t\,dt}_{=R}\right]$$

$$\Rightarrow R = \frac{1}{20} e^{20t} \sin 60t - \frac{3}{20} e^{20t} \cos 60t - 9R$$

$$\Rightarrow 10R = \frac{1}{20} e^{20t} \sin 60t - \frac{3}{20} e^{20t} \cos 60t$$

$$\Rightarrow R = \frac{1}{200} e^{20t} \sin 60t - \frac{3}{200} e^{20t} \cos 60t + C \quad \underline{\text{vital}}$$

<u>Recall</u>

$$e^{20t} I = 40R = 40\left(\frac{1}{200} e^{20t} \sin 60t - \frac{3}{200} e^{20t} \cos 60t + C\right)$$

$$\Rightarrow e^{20t} I = \frac{1}{5} e^{20t} \sin 60t - \frac{3}{5} e^{20t} \cos 60t + 40C$$

$$\Rightarrow I = \frac{1}{5} \sin 60t - \frac{3}{5} \cos 60t + 40C e^{-20t}$$

<u>IC</u>

$$I(0) = 1 = \frac{1}{5} \underbrace{\sin 0}_{0} - \frac{3}{5} \underbrace{\cos 0}_{1} + 40C \underbrace{e^{0}}_{1}$$

$$\Rightarrow 1 = -\frac{3}{5} + 40C$$

$$\Rightarrow \frac{8}{5} = 40C$$

$$\Rightarrow C = \frac{8}{200} = \frac{1}{25}$$

$$\therefore \boxed{I = \frac{1}{5} \sin 60t - \frac{3}{5} \cos 60t + \frac{8}{5} e^{-20t}}$$