

1.7 Linear Independence

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0, x_2 = 0, x_3 = 0$), but is this the *only solution*?

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in \mathbf{R}^n is said to be **linearly independent** if the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights c_1, \dots, c_p , not all 0, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0}.$$

↑
linear dependence relation
(when weights are not all zero)

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

- Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
- If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable \Rightarrow there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set

(b) Reduced echelon form:
$$\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = \\ x_2 = \\ x_3 \end{array}$$

Let $x_3 = \underline{\hspace{2cm}}$ (any nonzero number).

Then $x_1 = \underline{\hspace{2cm}}$ and $x_2 = \underline{\hspace{2cm}}$.

$$\underline{\hspace{2cm}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \underline{\hspace{2cm}} \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \underline{\hspace{2cm}} \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\underline{\hspace{2cm}} \mathbf{v}_1 + \underline{\hspace{2cm}} \mathbf{v}_2 + \underline{\hspace{2cm}} \mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Linear Independence of Matrix Columns

A linear dependence relation such as

$$-33 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. A Set of One Vector

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = \mathbf{0}$ is $x_1 = \underline{\hspace{2cm}}$.

So $\{\mathbf{v}_1\}$ is linearly independent when $\mathbf{v}_1 \neq \mathbf{0}$.

(b) Suppose

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = \text{---} \mathbf{v}_2$ if $c \neq 0$. But this is impossible since \mathbf{v}_1 is
_____ a multiple of \mathbf{v}_2 which means $c = \text{---}$.

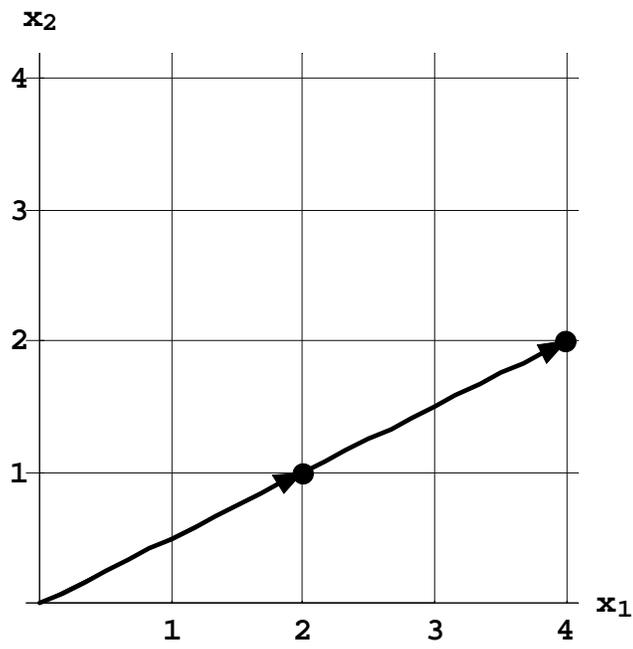
Similarly, $\mathbf{v}_2 = \text{---} \mathbf{v}_1$ if $d \neq 0$.

But this is impossible since \mathbf{v}_2 is not a multiple of \mathbf{v}_1 and so
 $d = 0$.

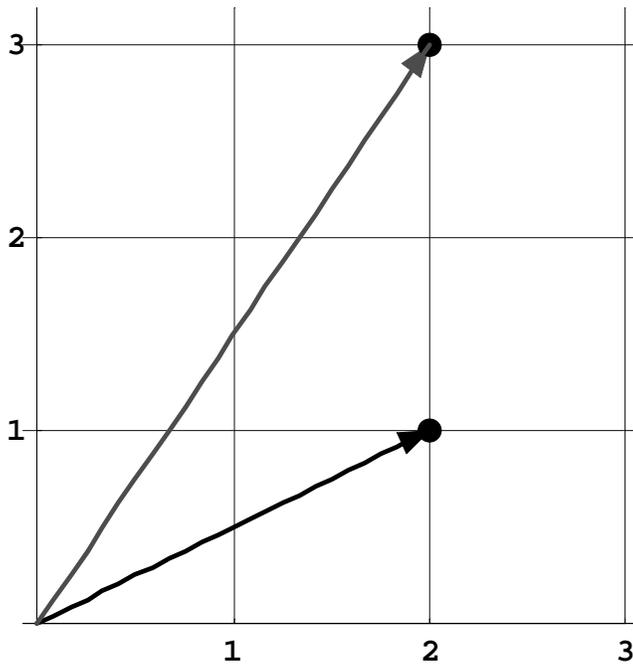
This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly _____ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly _____



linearly _____

3. A Set Containing the 0 Vector

Theorem 9

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in \mathbf{R}^n containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 = \mathbf{0}$. Then

$$1\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly dependent.

4. A Set Containing Too Many Vectors

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ in \mathbf{R}^n is linearly dependent if $p > n$.

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has more variables than equations

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has nontrivial solutions

\Rightarrow columns of A are linearly dependent

EXAMPLE With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

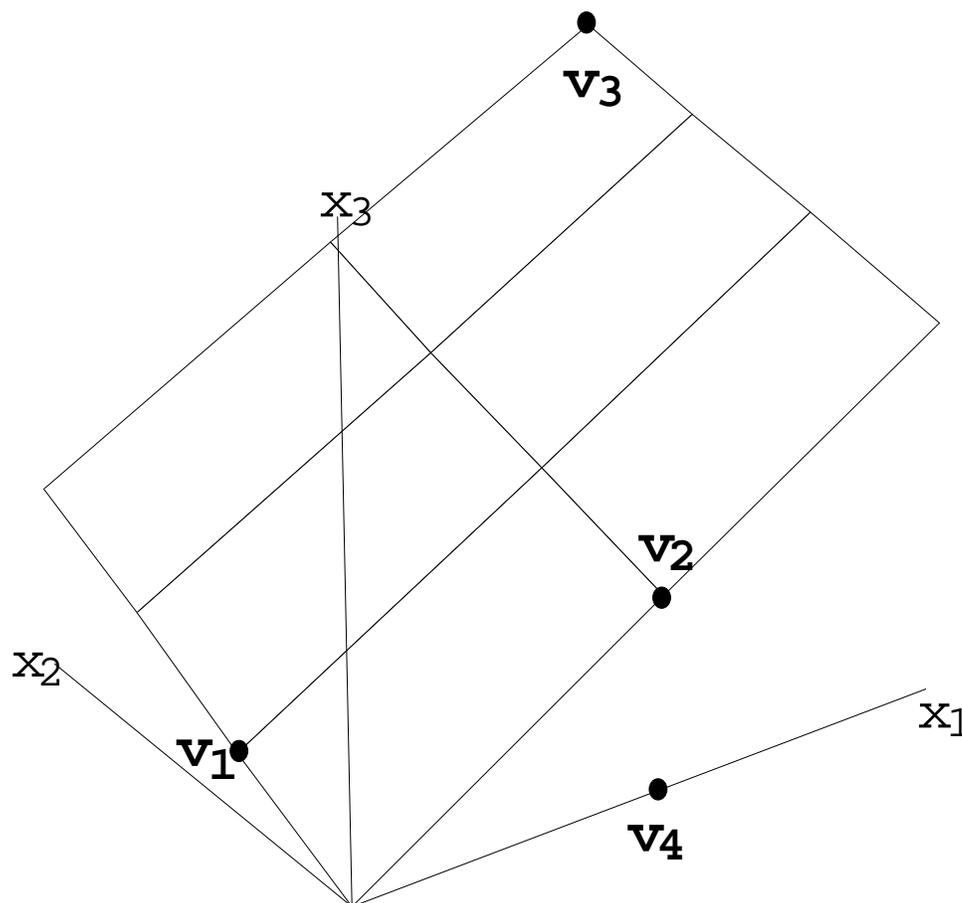
b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

$$\text{c. } \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{d. } \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Characterization of Linearly Dependent Sets

EXAMPLE Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in \mathbf{R}^3 in the following diagram. Is the set linearly dependent? Explain



Theorem 7

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector \mathbf{v}_j ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.