MATH 128 Calculus 2, Solutions to Term Test 1

[10] **1:** Let R be the region given by $0 \leq y \leq \dfrac{x}{\sqrt{4-x^2}}$ and $0 \leq x \leq 1$.

(a) Find the volume of the solid which is obtained by revolving R about the x-axis.

Solution: The volume is $V = \displaystyle\int_0^1 \frac{\pi\, x^2}{4-x^2}\, dx = \pi \int_0^1 -1 - \frac{4}{x^2-4}\, dx = \pi \int_0^1 -1 - \frac{1}{x-2} + \frac{1}{x+2}\, dx = \pi\Big[-x + \ln\left|\frac{x+2}{x-2}\right|\Big]_0^1 = \pi\big(-1+\ln 3\big)$

(b) Find the volume of the solid which is obtained by revolving R about the y-axis.

Solution: The volume is $V = \displaystyle\int_{x=0}^1 \frac{2\pi\, x^2\, dx}{\sqrt{4-x^2}}$. Let $2\sin\theta = x$ so that $2\cos\theta = \sqrt{4-x^2}$ and $2\cos\theta\, d\theta = dx$. Then $V = \displaystyle\int_{\theta=0}^{\pi/6} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{2\cos\theta} = 2\pi \int_0^{\pi/6} 4\sin^2\theta\, d\theta = 2\pi \int_0^{\pi/6} 2 - 2\cos 2\theta\ d\theta = 2\pi\Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} = 2\pi\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$.

[10] **2:** (a) Solve the initial value problem $2xy' = x + y + 1$ with $y(4) = 1$.

Solution: The DE is linear, since we can write it as $y' - \dfrac{1}{2x}\, y = \dfrac{x+1}{2x}$. An integrating factor is $\lambda = e^{\int -\frac{1}{2x}\, dx} = e^{-\frac{1}{2}\ln x} = x^{-1/2}$, and the solution to the DE is $y = x^{1/2} \displaystyle\int \frac{x+1}{2x^{3/2}}\, dx = x^{1/2} \int \tfrac{1}{2}\, x^{-1/2} + \tfrac{1}{2}\, x^{-3/2}\, dx = x^{1/2}\left(x^{1/2} - x^{-1/2} + c\right) = x - 1 + c\sqrt{x}$, for some constant c. To get $y(4) = 1$ we need $3 + 2c = 1$, so $c = -1$, and so the solution is $y = x - 1 - \sqrt{x}$.

(b) Solve the initial value problem $\sqrt{x^2+1}\, y' = x\,(y^2+1)$ with $y(0) = 0$.

Solution: This DE is separable. We write it as $\dfrac{dy}{1+y^2} = \dfrac{x\, dx}{\sqrt{1+x^2}}$ and integrate both sides to get $\tan^{-1} y = \sqrt{1+x^2} + c$ for some constant c. To get $y(0) = 0$ we need $0 = 1 + c$ so $c = -1$, and so the solution is given by $\tan^{-1} y = \sqrt{1+x^2} - 1$, that is $y = \tan\left(\sqrt{1+x^2} - 1\right)$.

[10] **3:** A tank initially contains 2 L of pure water. Over a period of two minutes, brine is poured into the tank at a rate of 1 L/min, the tank is kept well-mixed, and brine is drained from the tank at 1 L/min. The brine which is being poured into the tank has a concentration which varies with time: its concentration is given by $c(t) = (2-t)$ gm/L. Find the time at which the brine in the tank reaches its maximum concentration. (Hint: first find the amount of salt $S(t)$ in the tank at time t).

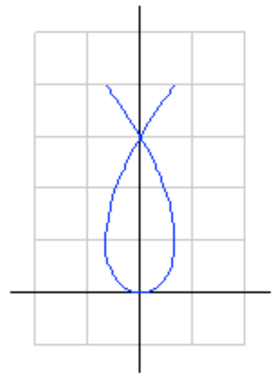
Solution: The volume V of brine in the tank remains constant at $V = 2$. The amount of salt $S(t)$ satisfies the DE $S' = r_1 c_1 - r_2 c_2$ where $r_1 = 1$, $c_1 = 2 - t$, $r_2 = 1$ and $c_2 = S/V = S/2$, that is $S' = 2 - t - \frac{1}{2}\, S$. This DE is linear since we can write it as $S' + \frac{1}{2}\, S = 2 - t$. An integrating factor is $\lambda = e^{t/2}$, and the solution is $S = e^{-t/2} \displaystyle\int (2-t)\, e^{t/2}\, dt$. We integrate by parts using $u = 2 - t$, $du = -dt$, $v = 2\, e^{t/2}$ and $dv = e^{t/2}$ to get $S = e^{-t/2}\left(2(2-t)\, e^{t/2} + \displaystyle\int 2\, e^{t/2}\right) = e^{-t/2}\Big((4-2t)\, e^{t/2} + 4\, e^{t/2} + a\Big) = 8 - 2t + a\, e^{-t/2}$ for some constant a. To get $S(0) = 0$ we need $8 + a = 0$ so $a = -8$, and so we have $S(t) = 8 - 2t - 8\, e^{t/2}$. The concentration of salt is given by $C(t) = S(t)/V = S(t)/2 = 4 - t - 4\, e^{-t/2}$. We have $C'(t) = -1 + 2\, e^{-t/2}$ so $C'(t) = 0 \Longrightarrow 2\, e^{-t/2} = 1 \Longrightarrow e^{-t/2} = \frac{1}{2} \Longrightarrow -t/2 = \ln\frac{1}{2} = -\ln 2 \Longrightarrow t = 2\ln 2$. Since $e^{-t/2}$ is a decreasing function, when $t < 2\ln 2$ we have $C'(t) > 0$ and when $t > 2\ln 2$ we have $C'(t) < 0$, so $C(t)$ is a maximum when $t = 2\ln 2$.

[10] **4:** Consider the parametric curve given by $(x,y) = \left(\frac{1}{3}t^3 - t, t^2\right)$ with $-2 \le t \le 2$.

(a) Sketch the curve, showing all intercepts and all horizontal and vertical points.

Solution: We have $x = 0 \iff \frac{1}{3}t^3 - t = 0 \iff \frac{1}{3}t(t-\sqrt{3})(t+\sqrt{3}) = 0 \iff t = 0$ or $\pm\sqrt{3}$, and $y = 0 \iff t^2 = 0 \iff t = 0$. Also $x' = t^2 - 1$ so $x' = 0 \iff t = \pm 1$ and $y' = 2t$ so $y' = 0 \iff t = 0$. Now we make a table of values and sketch the curve.

t	x	y
-2	$-\frac{2}{3}$	4
$-\sqrt{3}$	0	3
-1	$\frac{2}{3}$	1
0	0	0
1	$-\frac{2}{3}$	1
$\sqrt{3}$	0	3
2	$\frac{2}{3}$	4
3	6	9



(b) Find the arclength of the loop in the curve.

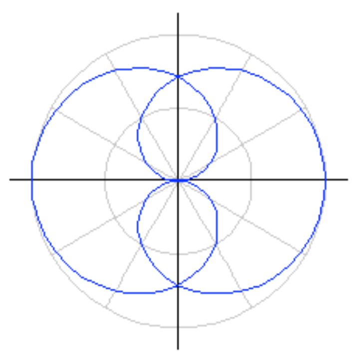
Solution: The arclength of the loop is $L = 2\int_0^{\sqrt{3}} \sqrt{x'(t)^2 + y'(t)^2}\,dt = 2\int_0^{\sqrt{3}} \sqrt{(t^2-1)^2 + (2t)^2}\,dt = 2\int_0^{\sqrt{3}} \sqrt{t^4 - 2t^2 + 1 + 4t^2}\,dt = 2\int_0^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\,dt = 2\int_0^{\sqrt{3}} \sqrt{(t^2+1)^2}\,dt = 2\int_0^{\sqrt{3}} t^2 + 1\;dt = 2\left[\frac{1}{3}t^3 + t\right]_0^{\sqrt{3}} = 4\sqrt{3}$.

[10] **5:** Consider the polar curve given by $r = 2\cos\frac{\theta}{2}$ with $0 \le \theta \le 4\pi$.

(a) Sketch the curve.

Solution: We make a table of values and sketch the curve.

θ	r
0	2
$\pi/3$	$\sqrt{3}$
$\pi/2$	$\sqrt{2}$
$2\pi/3$	1
π	0
$4\pi/3$	-1
$3\pi/2$	$-\sqrt{2}$
$5\pi/3$	$-\sqrt{3}$
2π	-2



(b) Find the area of one inner loop of the curve.

Solution: The area of one inner loop is $L = 2\int_{\pi/2}^{\pi} \frac{1}{2}\left(2\cos\frac{\theta}{2}\right)^2\,d\theta = \int_{\pi/2}^{\pi} 4\cos^2\frac{\theta}{2}\,d\theta = \int_{\pi/2}^{\pi} 2 + 2\cos\theta\;d\theta = \Big[2\theta + 2\sin\theta\Big]_{\pi/2}^{\pi} = 2\pi - (\pi + 2) = \pi - 2$.