

1.8 Introduction to Linear Transformations

Another way to view $A\mathbf{x} = \mathbf{b}$:

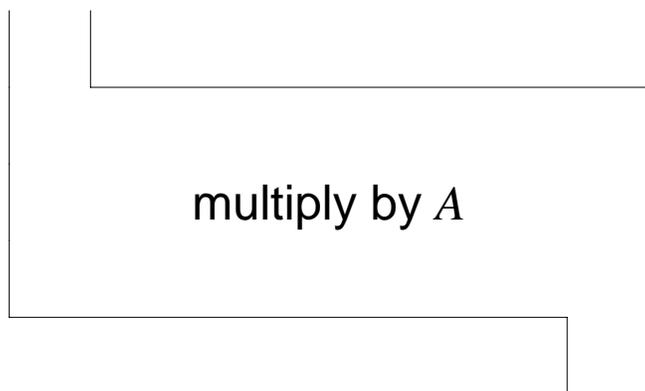
Matrix A is an object acting on \mathbf{x} by multiplication to produce a new vector $A\mathbf{x}$ or \mathbf{b} .

EXAMPLE:

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

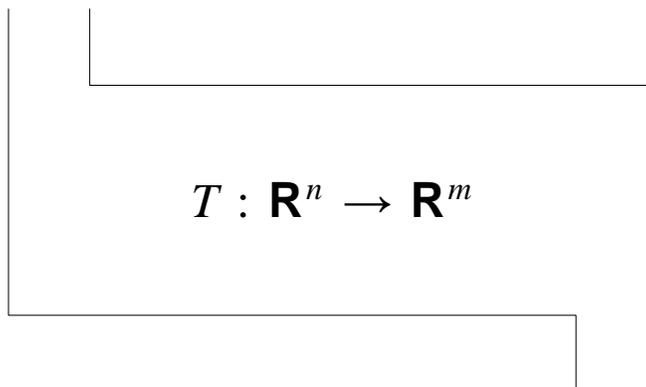
Suppose A is $m \times n$. Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all _____ in \mathbf{R}^n which are transformed into vector \mathbf{b} in \mathbf{R}^m through multiplication by A .



transformation
“machine”

Matrix Transformations

A **transformation** T from \mathbf{R}^n to \mathbf{R}^m is a rule that assigns to each vector \mathbf{x} in \mathbf{R}^n a vector $T(\mathbf{x})$ in \mathbf{R}^m .



Terminology:

\mathbf{R}^n : **domain** of T

\mathbf{R}^m : **codomain** of T

$T(\mathbf{x})$ in \mathbf{R}^m is the **image** of \mathbf{x} under the transformation T

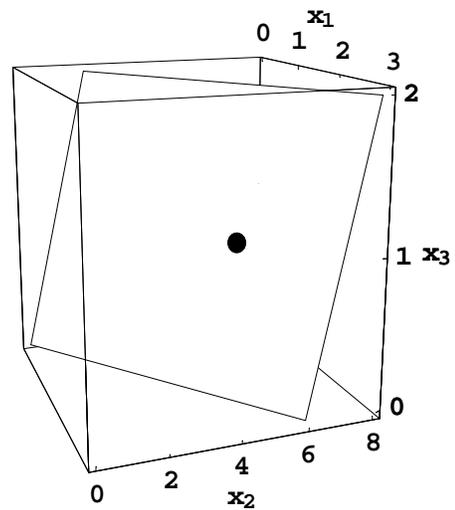
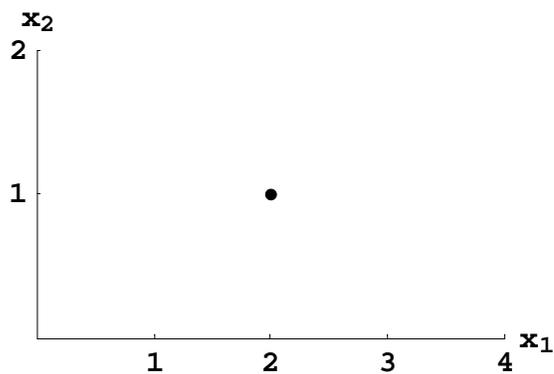
Set of all images $T(\mathbf{x})$ is the **range** of T

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation

$T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$



EXAMPLE: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$,

$\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then define a transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

- Find an \mathbf{x} in \mathbf{R}^3 whose image under T is \mathbf{b} .
- Is there more than one \mathbf{x} under T whose image is \mathbf{b} .
(*uniqueness problem*)
- Determine if \mathbf{c} is in the range of the transformation T .
(*existence problem*)

Solution: (a) Solve _____ = _____ for \mathbf{x} .

I.e., solve _____ = _____ or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Augmented matrix:

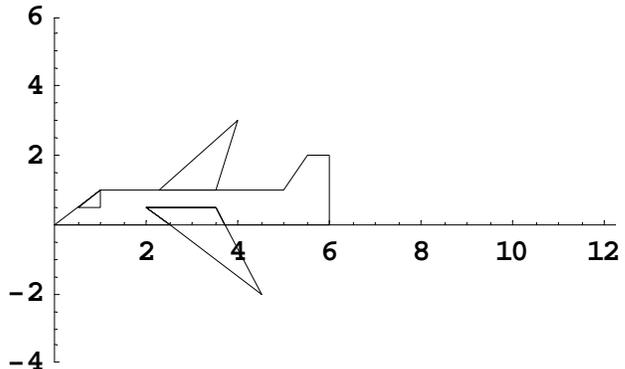
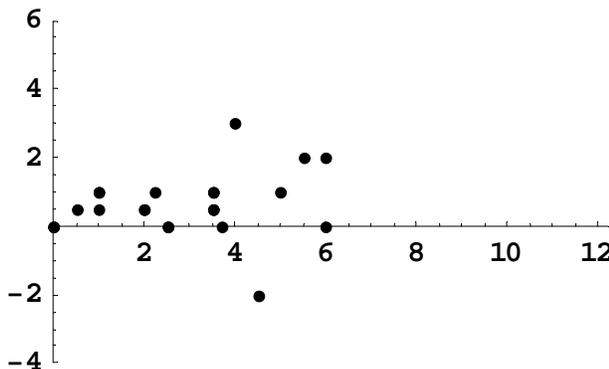
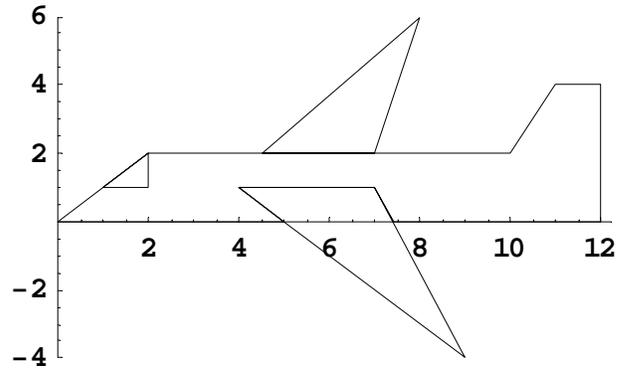
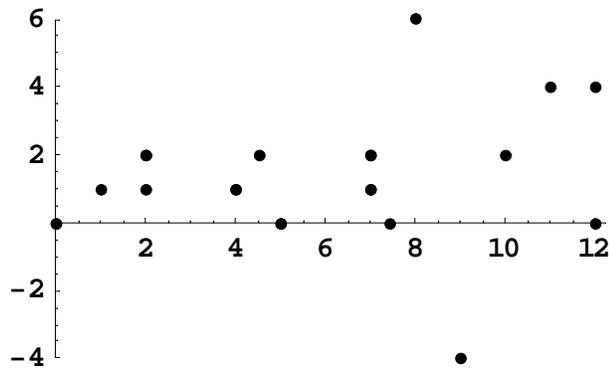
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Matrix transformations have many applications - including *computer graphics*.

EXAMPLE: Let $A = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$. The transformation

$T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is an example of a **contraction** transformation. The transformation $T(\mathbf{x}) = A\mathbf{x}$ can be used to move a point \mathbf{x} .

$$\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \quad T(\mathbf{u}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Linear Transformations

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= A(\mathbf{u} + \mathbf{v}) = \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \\ &= \underline{\hspace{2cm}} + \underline{\hspace{2cm}} \end{aligned}$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \underline{\hspace{1cm}}A\mathbf{u} = \underline{\hspace{1cm}}T(\mathbf{u})$$

for all \mathbf{u}, \mathbf{v} in \mathbf{R}^n and all scalars c .

DEFINITION

A transformation T is **linear** if:

- i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in the domain of T .
- ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all \mathbf{u} in the domain of T and all scalars c .

Every matrix transformation is a **linear** transformation.

RESULT If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

Proof:

$$T(\mathbf{0}) = T(0\mathbf{u}) = \underline{\quad}T(\mathbf{u}) = \underline{\quad}.$$

$$T(c\mathbf{u} + d\mathbf{v}) = T(\quad) + T(\quad)$$

$$= \underline{\quad}T(\quad) + \underline{\quad}T(\quad)$$

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and

$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ is a linear transformation

which maps \mathbf{e}_1 into \mathbf{y}_1 and \mathbf{e}_2 into \mathbf{y}_2 . Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$$T(\mathbf{e}_1) = \underline{\hspace{2cm}} \quad \text{and} \quad T(\mathbf{e}_2) = \underline{\hspace{2cm}}.$$

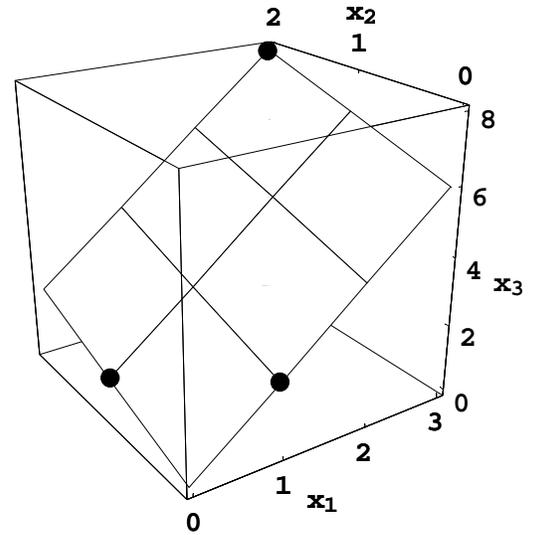
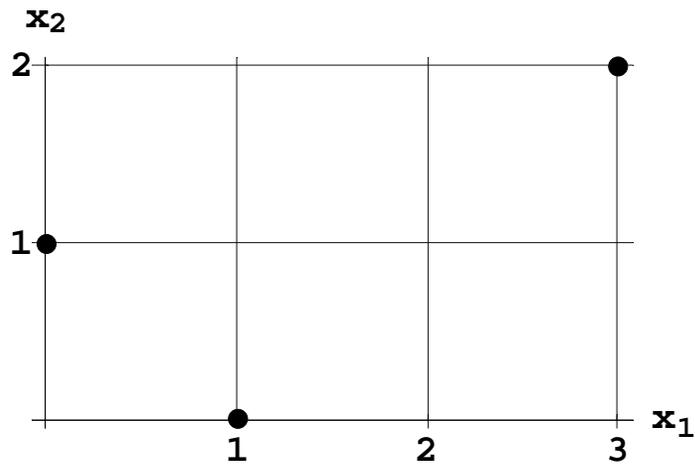
Also

$$\underline{\hspace{2cm}}\mathbf{e}_1 + \underline{\hspace{2cm}}\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(\underline{\hspace{2cm}}\mathbf{e}_1 + \underline{\hspace{2cm}}\mathbf{e}_2) =$$

$$\underline{\hspace{2cm}}T(\mathbf{e}_1) + \underline{\hspace{2cm}}T(\mathbf{e}_2) =$$



$$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$$

Also

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\underline{\hspace{1cm}}\mathbf{e}_1 + \underline{\hspace{1cm}}\mathbf{e}_2) =$$

$$\underline{\hspace{1cm}}T(\mathbf{e}_1) + \underline{\hspace{1cm}}T(\mathbf{e}_2) =$$

EXAMPLE: Define $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ such that $T(x_1, x_2, x_3) = (|x_1 + x_3|, 2 + 5x_2)$. Show that T is not a linear transformation.

Solution: Another way to write the transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Provide a **counterexample** - example where $T(\mathbf{0}) = \mathbf{0}$, $T(c\mathbf{u}) = cT(\mathbf{u})$ or $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is violated.

A counterexample:

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \\ 2 + 5 \cdot 0 \end{bmatrix} \neq \underline{\hspace{2cm}}$$

which means that T is not linear.

Another counterexample: Let $c = -1$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} |-1 + -1| \\ 2 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and

$$cT(\mathbf{u}) = -1T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}.$$

Therefore $T(c\mathbf{u}) \neq \text{---}T(\mathbf{u})$ and therefore T is

not _____.