MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 8 SOLUTIONS

Due **Wednesday, November 22** in Drop Box 9 before class
To receive full marks, correct answers must be fully justified.

1. Manipulate a known power series (from the list at the bottom of pages 612 and 618 of your textbook) to find a Maclaurin series for the following functions. Determine the radius of convergence of each series.

   (a) $f(x) = \dfrac{x^2}{(1-2x)^2}$

   Solution 1 Notice that $f(x)$ is the derivative of the sum of a geometric series multiplied by $\dfrac{x^2}{2}$:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{1-2x}\right) &= \frac{d}{dx}(1-2x)^{-1} = -(1-2x)^{-2}(-2) = \frac{2}{(1-2x)^2} = \frac{2}{x^2}f(x) \\
\Rightarrow f(x) &= \frac{x^2}{2}\left[\frac{d}{dx}\left(\frac{1}{1-2x}\right)\right] = \frac{x^2}{2}\left(\frac{d}{dx}\sum_{n=0}^{\infty}(2x)^n\right) = \frac{x^2}{2}\left(\frac{d}{dx}\sum_{n=0}^{\infty}2^n x^n\right) \\
&= \frac{x^2}{2}\sum_{n=0}^{\infty}2^n n x^{n-1} = \sum_{n=0}^{\infty}2^{n-1}nx^{n-1+2} = \sum_{n=0}^{\infty}2^{n-1}nx^{n+1}
\end{aligned}$$

   with radius of convergence $R = \frac{1}{2}$, since the radius of convergence of $\sum_{n=0}^{\infty}(2x)^n$ is $\frac{1}{2}$.

   ASIDE: Although a correct answer, since the $n = 0$ term will be 0, we can omit this index value to obtain:

$$f(x) = \sum_{n=1}^{\infty}2^{n-1}nx^{n+1}$$

   and to make the index start at 0 now, we can shift it back to 0 to obtain:

$$f(x) = \sum_{n=0}^{\infty}2^n(n+1)x^{n+2}$$

   so that we see it is equivalent to the result in Solution 2.

   Solution 2 We can express $f(x)$ as the product of x^2 and the binomial series for $\dfrac{1}{(1-2x)^2}$ which converges when $|-2x| < 1 \Rightarrow |x| < \frac{1}{2}$ so it has radius of convergence $R = \frac{1}{2}$:

$$\begin{aligned}
\frac{1}{(1-2x)^2} &= (1-2x)^{-2} = \sum_{n=0}^{\infty}\binom{-2}{n}(-2x)^n = \sum_{n=0}^{\infty}\binom{-2}{n}(-1)^n 2^n x^n \\
&= \sum_{n=0}^{\infty}\frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!}(-1)^n 2^n x^n = \sum_{n=0}^{\infty}\frac{(-1)^n[(2)(3)(4)\cdots(n+1)]}{n!}(-1)^n 2^n x^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n[n!(n+1)]}{n!}(-1)^n 2^n x^n = \sum_{n=0}^{\infty}(n+1)2^n x^n \\
\Rightarrow f(x) &= x^2\sum_{n=0}^{\infty}(n+1)2^n x^n = \sum_{n=0}^{\infty}(n+1)2^n x^{n+2}, \text{ with radius of convergence } R = \tfrac{1}{2}.
\end{aligned}$$

(b) $S(x) = \displaystyle\int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$ (Note of interest: $S(x)$ is the Fresnel function. See page 380 of your textbook.)

Solution We will integrate a form of the Maclaurin series for $\sin t$ to express $S(x)$ as a Maclaurin series:

$$\begin{aligned}
\sin t &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \;\Rightarrow\; \sin\left(\frac{\pi t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi t^2}{2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1} t^{4n+2}}{(2n+1)!}, R = \infty \\
\Rightarrow S(x) &= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1} t^{4n+2}}{(2n+1)!}\right] dt = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!} \cdot \frac{t^{4n+3}}{4n+3}\Bigg]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1} x^{4n+3}}{(2n+1)!(4n+3)}, R = \infty
\end{aligned}$$

2. Find the Taylor series centered at $x = \pi$ for $f(x) = \cos^2 x$.

Solution 1 Since $f(x) = \cos^2 x = \dfrac{1 + \cos 2x}{2}$ and since $\cos A = \cos(A \pm 2\pi)$, then

$$\cos 2x = \cos(2x - 2\pi) = \cos(2(x-\pi)) \;\Rightarrow\; f(x) = \frac{1 + \cos(2(x-\pi))}{2} = \frac{1}{2}[1 + \cos(2(x-\pi))].$$

Then we can use the known Maclaurin series for $\cos x$ to obtain:

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2(x-\pi))^{2n}}{(2n)!}\right) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} (x-\pi)^{2n}}{(2n)!} \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!}(x-\pi)^{2n}
\end{aligned}$$

or, extracting the $n = 0$ term and adding it to the $\frac{1}{2}$ we have:

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!}(x-\pi)^{2n}$$

Either answer is fine.

Solution 2 Compute the coefficients of the Taylor series directly:

$$\left.\begin{array}{ll}
f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x & f(\pi) = 1 \\
f'(x) = -\sin(2x) & f'(\pi) = 0 \\
f''(x) = -2\cos(2x) & f''(\pi) = -2 \\
f'''(x) = 4\sin(2x) & f'''(\pi) = 0 \\
f^{(iv)}(x) = 8\cos(2x) & f^{(iv)}(\pi) = 8 \\
f^{(v)}(x) = -16\sin(2x) & f^{(v)}(\pi) = 0 \\
f^{(vi)}(x) = -32\cos(2x) & f^{(vi)}(\pi) = -32 \\
etc. & etc.
\end{array}\right\} \Rightarrow \begin{aligned} f(x) &= 1 - \tfrac{2}{2!}(x-\pi)^2 + \tfrac{8}{4!}(x-\pi)^4 - \tfrac{32}{6!}(x-\pi)^6 + \ldots \\ &= 1 - (x-\pi)^2 + \tfrac{1}{3}(x-\pi)^4 - \tfrac{2}{45}(x-\pi)^6 + \ldots \end{aligned}$$

Although this answer is correct, we can write it using summation notation by noting the pattern in the coefficients:

Clearly, the coefficients of odd powers of $(x - \pi)$ will be 0. The even derivatives evaluated at $x = \pi$ alternate in sign and are powers of 2: $2^0 = 1, -2^1 = -2, 2^3 = 8, -2^5 = -32$, *etc.* Hence $f^{(2n)}(\pi) = (-1)^n 2^{2n-1}, n \geq 1$, where $f^{(0)}(\pi) = f(\pi) = 1$. Thus we can write

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!}(x-\pi)^{2n}.$$

Solution 3 The Taylor series for $\cos x$ centered at $x = \pi$ can be multiplied by itself and simplified by collecting like terms. This approach is tedious and not shown here.

3. Find the Taylor polynomial of degree 3 centered at $x = 1$ for $f(x) = \ln(1 + x^2)$.

   Solution Compute the coefficients of the Taylor series directly:

$$\left.\begin{aligned} f(x) &= \ln(1+x^2) & f(1) &= \ln 2 \\ f'(x) &= \frac{2x}{1+x^2} & f(1) &= 1 \\ f''(x) &= \frac{2-2x^2}{(1+x^2)^2} & f''(1) &= 0 \\ f'''(x) &= \frac{4x(x^2-3)}{(1+x^2)^3} & f'''(1) &= -1 \end{aligned}\right\} \Rightarrow f(x) \quad \approx \ln 2 + (x-1) - \tfrac{(x-1)^3}{3!}.$$

4. Consider the polar curve $r = -\cos t$ for $\pi \le t \le 2\pi$.

   (a) Find all of the angles t where the curve has vertical tangents.

   Solution With $x = r\cos t = -\cos^2 t$ we have $\dfrac{dx}{dt} = -2\cos t(-\sin t) = 2\cos t \sin t$. When $\dfrac{dx}{dt} = 0$ the curve has vertical tangents, that is when $\cos t = 0 \Rightarrow t = \frac{3\pi}{2}$ or when $\sin t = 0 \Rightarrow t = \pi, 2\pi$ (due to the given restriction $\pi \le t \le 2\pi$).

   (b) Find all of the angles t where the curve has horizontal tangents.

   Solution With $y = r\sin t = -\cos t \sin t$ we have $\dfrac{dy}{dt} = \sin t \cos t - \cos t \cos t = \sin^2 t - \cos^2 t$. When $\dfrac{dy}{dt} = 0$ the curve has horizontal tangents, that is when $\sin^2 t - \cos^2 t = 0$. We can deduce that $t = \frac{5\pi}{4}, \frac{7\pi}{4}$ in two different ways:

   (i) $\sin^2 t - \cos^2 t = \cos 2t = 0 \Rightarrow t = \frac{5\pi}{4}, \frac{7\pi}{4}$, or

   (ii) $\sin^2 t - \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \tan^2 t = 1 \Rightarrow \tan t = \pm 1$ where $\tan t = 1 \Rightarrow t = \frac{5\pi}{4}$ and $\tan t = -1 \Rightarrow t = \frac{7\pi}{4}$,

   again keeping the given restriction on t in mind.

   (c) Find the Cartesian equation of the curve.

   Solution With $x = r\cos t$ we have $\cos t = \frac{x}{r}$ so our polar equation $r = -\cos t$ becomes $r = -\frac{x}{r}$ or $r^2 = -x$. Using $r^2 = x^2 + y^2$ we have $-x = x^2 + y^2$.

   While this answer is correct, it does not reveal the type of curve we have. Completing the square we have:

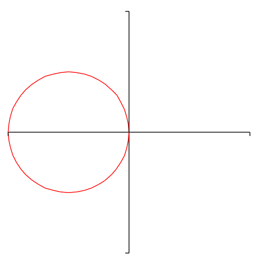
$$x^2 + x + y^2 = 0 \Rightarrow x^2 + x + \frac{1}{4} - \frac{1}{4} + y^2 = 0 \Rightarrow \left(x + \frac{1}{2}\right)^2 + y^2 = \frac{1}{4},$$

   which is a circle of radius $\frac{1}{2}$ centered at $(x, y) = \left(-\frac{1}{2}, 0\right)$.

   (d) Sketch the curve. Include a table of values (t, r) that justify your curve.

   Solution

t	$r = -\cos t$
π	1
$\frac{7\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$\frac{1}{2}$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$-\frac{1}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{\sqrt{3}}{2}$
2π	-1

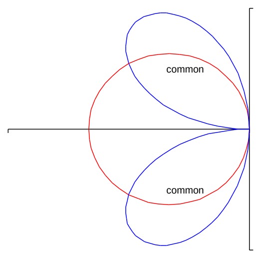


   Note it is important to check the restriction on t rather than assume the Cartesian equation represents the polar curve entirely. Here, coincidentally, it does. However, if we had begun with a different restriction on t, *e.g.* $\pi \le t \le \frac{3\pi}{2}$, the polar curve would resemble only part of the circle (its bottom half).

5. Find the area of the region that lies inside both curves $r = -\cos\theta$ and $r = \sin 2\theta$ for $\pi \le \theta \le 2\pi$. (That is, find the area of the region that both curves have in common.)

Solution First we sketch both curves on the same polar grid. The curve $r = \cos t$ has been plotted in a previous problem here. The second one is obtained by plotting points as shown.

t	$r = \sin(2t)$
π	0
$\frac{7}{6}\pi$	$\frac{\sqrt{3}}{2}$
$\frac{5}{4}\pi$	1
$\frac{4}{3}\pi$	$\frac{\sqrt{3}}{2}$
$\frac{3}{2}\pi$	0
$\frac{5}{3}\pi$	$-\frac{\sqrt{3}}{2}$
$\frac{7}{4}\pi$	-1
$\frac{11}{6}\pi$	$-\frac{\sqrt{3}}{2}$
2π	0



The sketch is needed to help set up the integral for the common areas of the two curves.

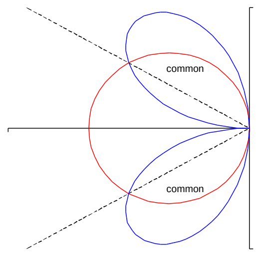
The curves intersect when $-\cos t = \sin(2t) = 2\sin t\cos t \Rightarrow -1 = 2\sin t$ such that $\cos t \neq 0$ (since we have divided by $\cos t$).

$$\sin t = -\frac{1}{2} \Rightarrow t = \frac{7\pi}{6}, \frac{11\pi}{6}$$

The special case of $\cos t = 0$ (which we isolated earlier) implies that $t = \dfrac{3\pi}{2}$.

The total common area A can be calculated two ways:

(i) Using symmetry, we can consider the top common area and multiply it by two to get the total area. Furthermore, we must break up the common area into two parts, since the region is not bounded by two curves in a simple way. We break up the common area along the dashed line in the following figure and set up an integral for each section on either side of the dashed line.



$$\begin{aligned}
A &= 2\int_{3\pi/2}^{11\pi/6} \frac{1}{2}(-\cos t)^2\,dt + 2\int_{11\pi/6}^{2\pi} \frac{1}{2}(\sin(2t))^2\,dt = \int_{3\pi/2}^{11\pi/6} \frac{1}{2} + \frac{\cos(2t)}{2}\,dt + \int_{11\pi/6}^{2\pi} \frac{1}{2} - \frac{\cos(4t)}{2}\,dt \\
&= \left[\frac{t}{2} + \frac{\sin(2t)}{4}\right]_{3\pi/2}^{11\pi/6} + \left[\frac{t}{2} - \frac{\sin(4t)}{8}\right]_{11\pi/6}^{2\pi} \\
&= \frac{11\pi}{12} + \frac{\sin\left(\frac{11\pi}{3}\right)}{4} - \frac{3\pi}{4} - \frac{\sin(3\pi)}{4} + \pi - \frac{\sin(8\pi)}{8} - \frac{11\pi}{12} + \frac{\sin(\frac{22\pi}{3})}{8} \\
&= \frac{\pi}{4} - \frac{3\sqrt{3}}{16}
\end{aligned}$$

(ii) The second method involves seeing A as the difference of areas:

$$\begin{aligned}
A &= \text{(area of the entire circle)} - \text{(area of region to left of the loops but inside the circle)} \\
&= \frac{\pi}{4} - 2\int_{11\pi/6}^{2\pi} \frac{1}{2}\left((\cos t)^2 - (\sin(2t))^2\right)\, dt \\
&= \frac{\pi}{4} - \int_{11\pi/6}^{2\pi} \frac{1}{2} + \frac{\cos(2t)}{2} - \frac{1}{2} + \frac{\cos(4t)}{2}\, dt \\
&= \frac{\pi}{4} - \left[\frac{\sin(2t)}{4} + \frac{\sin(4t)}{8}\right]_{11\pi/6}^{2\pi} \\
&= \frac{\pi}{4} - \left(\frac{\sin(4\pi)}{4} + \frac{\sin(8\pi)}{8} - \frac{\sin(11\pi/3)}{4} - \frac{\sin(22\pi/3)}{8}\right) \\
&= \frac{\pi}{4} - \frac{3\sqrt{3}}{16}
\end{aligned}$$

where we have used symmetry in the integral.

Useful trigonometric identities:

$$\sin 2t = 2\sin t\cos t, \quad \cos 2t = \cos^2 t - \sin^2 t, \quad \sin^2 t = \frac{1 - \cos 2t}{2}, \quad \cos^2 t = \frac{1 + \cos 2t}{2}.$$

Refer to Appendix C for trigonometric function values for specific angles.