MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 6 SOLUTIONS

Due **Wednesday, November 1** in Drop Box 9 before class

To receive full marks, correct answers must be fully justified.

1. Find the limit of each of the following sequences a_n, if the limit exists.
   (Unless otherwise stated, assume that $n \geq 1$, where n is a positive integer: $1, 2, 3, \ldots$.)

   (a) $a_n = \dfrac{\sqrt{n^2-6}}{5+\sqrt{4n^2+3n+2}}$, $n \geq 3$

   Solution: $a_n = \frac{\sqrt{n^2-6}}{5+\sqrt{4n^2+3n+2}} \frac{1/n}{1/n} = \frac{\sqrt{(1-\frac{6}{n^2})}}{\frac{5}{n}+\sqrt{(4+\frac{3}{n}+\frac{2}{n^2}}} \to \frac{1}{\sqrt{4}} = \frac{1}{2}$ as $n \to \infty$.
   Therefore, the sequence is convergent.

   (b) $a_n = \dfrac{n!}{3^n}$

   Solution: $a_n = \frac{n!}{3^n} = \frac{1\cdot2\cdot3\cdots n}{3\cdot3\cdot3\cdots3} = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3}{3}\cdot\frac{4}{3}\cdots\frac{n-1}{3}\cdot\frac{n}{3} = \frac{1}{3}\cdot\frac{2}{3}\cdot$ (product of $n-3$ terms, each ≥ 1) $\cdot\frac{n}{3}$
   $\Rightarrow a_n \geq \frac{1}{3}\cdot\frac{2}{3}\cdot(1)\cdot\frac{n}{3} = \frac{2n}{27} \to \infty$ as $n \to \infty$. Therefore, the sequence is divergent.

   (c) $a_1 = 0, a_{n+1} = 1 - a_n$ (Hint: calculate some terms of this recursive sequence.)

   Solution: $\{a_n\} = \{0, 1, 0, 1, 0, \ldots\}$, which has no limit, since the terms alternate repeatedly between 0 and 1, so the sequence is divergent.

   (d) $a_n = \dfrac{(\ln n)^2}{n}$

   Solution: Let $f(x) = \frac{(\ln x)^2}{x}$ and consider $x \to \infty$:

$$
\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{(\ln x)^2}{x} \leftarrow \frac{\infty}{\infty} \text{ type, use l'Hospital's rule (H):} \\
&\stackrel{H}{=} \lim_{x\to\infty} \frac{2\ln x \frac{1}{x}}{1} = 2\lim_{x\to\infty}\frac{\ln x}{x} \ \leftarrow \frac{\infty}{\infty} \text{ type} \\
&\stackrel{H}{=} 2\lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.
\end{aligned}
$$

   Therefore, the sequence is convergent.

   (e) $a_n = \dfrac{(-10)^n}{3^{2n+1}}$

   Solution: $a_n = \frac{(-10)^n}{3^{2n+1}} = \frac{(-10)^n}{3^{2n}3} = \frac{1}{3}\frac{(-10)^n}{9^n} = \frac{1}{3}\left(-\frac{10}{9}\right)^n = \frac{(-1)^n}{3}\left(\frac{10}{9}\right)^n \to \infty$ as $n \to \infty$.
   Therefore, the sequence is divergent.

2. Let $a_1 = \frac{5}{2}$ and $a_{n+1} = \dfrac{{a_n}^2+1}{2}$, $n \geq 1$. Determine whether $\{a_n\}$ converges, and if so, then find the limit.

   Solution: The first few terms are $a_1 = 5/2, a_2 = 29/8$, and $a_3 = 905/128$. Since the sequence appears to be increasing, we shall try to prove using mathematical induction that $a_n < a_{n+1}$ for all $n \geq 1$.

   Proof that $a_n < a_{n+1}$ for all $n \geq 1$:

   (1) Check $n = 1$: $a_1 = 5/2 < 29/8 = a_2$.

   (2) Assume true for $n = k$: $a_k < a_{k+1}$.

   (3) Prove true for $n = k+1$: $a_k < a_{k+1} \Rightarrow {a_k}^2 < {a_{k+1}}^2 \Rightarrow {a_k}^2 + 1 < {a_{k+1}}^2 + 1 \Rightarrow \frac{{a_k}^2+1}{2} < \frac{{a_{k+1}}^2+1}{2} \Rightarrow a_{k+1} < a_{k+2}$, (based on the definition of a_{n+1}). So, we have proved that $a_n < a_{n+1}$ for $n = k+1$.

   Thus, by mathematical induction, we have proved that $a_n < a_{n+1}$ is true for all n. (That is, we have proved that the sequence is increasing for all n.)

   Convergence:

   We have yet to determine whether the sequence is convergent. (So far, we know that it is increasing from 5/2. But is the sequence bounded by some value, L?)

   If the sequence is convergent, then all terms in the sequence must approach the same limiting value, L, as $n \to \infty$. If so, then we can say that $a_n \to L$ and also that $a_{n+1} \to L$. Using the definition of the sequence we have:

   $$L = \frac{L^2+1}{2} \Rightarrow 2L = L^2 + 1 \Rightarrow L^2 - 2L + 1 = (L-1)^2 = 0 \Rightarrow L = 1.$$

   We have arrived at a contradiction, since our mathematical induction proof revealed that the sequence is increasing for all n from an initial value of $5/2 = 2.5 > 1 = L$.

   Therefore, the sequence increases without bound. Thus the sequence is divergent.

3. Let $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2 + x_n}, n \geq 1$. Determine whether $\{x_n\}$ converges, and if so, then find the limit.

   Solution: The first few terms of the sequence are $x_1 = \sqrt{2}, x_2 = \sqrt{2+\sqrt{2}}$, and $x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$.

   It appears that the terms in the sequence are increasing.

Aside: We can see that $x_2 > x_1$ since $2 + \sqrt{2} > 2$ and, since $\sqrt{\cdot}$ is an increasing function, then $\sqrt{2+\sqrt{2}} > \sqrt{2}$. Likewise, $x_3 > x_2$ since $2 + \sqrt{2+\sqrt{2}} > 2 + \sqrt{2} \Rightarrow \sqrt{2+\sqrt{2+\sqrt{2}}} > \sqrt{2+\sqrt{2}}$.

   Since the sequence appears to be increasing, we shall try to prove that $x_n < x_{n+1}$ for all $n \geq 1$ using mathematical induction.

   Proof that $x_n < x_{n+1}$ for all $n \geq 1$:

   (1) Check $n = 1$: $x_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = x_2$

   (2) Assume true for $n = k$: $x_k < x_{k+1}$.

   (3) Prove true for $n = k+1$: $x_k < x_{k+1} \Rightarrow x_k + 2 < x_{k+1} + 2 \Rightarrow \sqrt{x_k + 2} < \sqrt{x_{k+1} + 2} \Rightarrow x_{k+1} < x_{k+2}$, (based on the definition of x_{n+1}). So, we have proved that $x_n < x_{n+1}$ for $n = k+1$.

   Thus, by mathematical induction, we have proved that $x_n < x_{n+1}$ is true for all n. (That is, we have proved that the sequence is increasing for all n.)

Convergence:

We have yet to determine whether the sequence is convergent. (So far, we know that it is increasing from $\sqrt{2}$. But is the sequence bounded by some value, L?)

If the sequence is convergent, then all terms in the sequence must approach the same limiting value, L, as $n \to \infty$. If so, then we can say that $x_n \to L$ and also that $x_{n+1} \to L$. Using the definition of the sequence we have:

$$L = \sqrt{2+L} \Rightarrow L^2 = 2 + L \Rightarrow L^2 - L - 2 = (L+1)(L-2) = 0 \Rightarrow L = -1 \text{ or } L = 2.$$

Since $L > 0$ (the sequence terms are increasing from an initial value of $\sqrt{2}$), the limit must be 2.

4. Find the sum of the following series, if the sum exists. If the series is divergent, clearly justify your answer.

   (a) $\sum_{n=1}^{\infty} \sin n$

   Solution: Since $\lim_{n\to\infty} \sin n$ does not exist, the series is divergent, by the Test for Divergence.

   (b) $\sum_{n=1}^{\infty} \frac{1+3^n}{2^{n+1}}$

   Solution:

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{2^n 2} + \sum_{n=1}^{\infty} \frac{3^n}{2^n 2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$$

   where the first series is a convergent geometric series ($r = 1/2$) but the second series is a divergent geometric series ($r = 3/2$). Hence the given series is divergent.

   (c) $\sum_{n=1}^{\infty} \frac{1-3^{2-n}}{2^{n+2}}$

   Solution:

$$\sum_{n=1}^{\infty} \frac{1-3^{2-n}}{2^{n+2}} = \sum_{n=1}^{\infty} \frac{1}{2^n 2^2} - \sum_{n=1}^{\infty} \frac{3^2 3^{-n}}{2^n 2^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{2^n} - \frac{9}{4}\sum_{n=1}^{\infty} \frac{1}{6^n}$$

   where both series are convergent geometric series: the first with $r = 1/2$, the second with $r = 1/6$.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1-3^{2-n}}{2^{n+2}} = \frac{1}{4}\left(\frac{1/2}{1-1/2}\right) - \frac{9}{4}\left(\frac{1/6}{1-1/6}\right) = 1/4 - 9/20 = -1/5$$

   (d) $\sum_{n=1}^{\infty} \frac{2n^3}{n^2+4n+3}$

   Solution:

   We attempt the Test for Divergence first: $\lim_{n\to\infty} \frac{2n^3}{n^2+4n+3} = \infty$, therefore, the series is divergent.

(e) $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

Solution: Using partial fractions, we express the N^{th} partial sum as

$$\begin{aligned}
S_N = \sum_{n=1}^{N} \frac{2}{n^2+4n+3} &= \sum_{n=1}^{N} \frac{A}{n+1} + \frac{B}{n+3} = \sum_{n=1}^{N} \frac{An+3A+Bn+B}{n^2+4n+3} \\
\Rightarrow 2 = An + 3A + Bn + B &= n(A+B) + 3A + B \Rightarrow \{2 = 3A + B, 0 = A + B\} \\
\Rightarrow A = -B \Rightarrow 2 &= -3B + B = -2B \Rightarrow B = -1 \Rightarrow A = 1 \\
\Rightarrow S_N &= \sum_{n=1}^{N} \frac{1}{n+1} - \frac{1}{n+3} \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\
&\quad + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+2}\right) + \left(\frac{1}{N+1} - \frac{1}{N+3}\right) \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3}
\end{aligned}$$

(a telescoping series; many terms cancel).

Therefore, $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+2} - \frac{1}{N+3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Thus, the series is convergent.

5. When money is spent on goods and services, those that receive the money also spend some of it. The people receiving some of that twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*.

   Suppose that nationwide, approximately 92% of all income is spent and 8% is saved. What is the total amount of spending generated by a 50 billion dollar tax rebate if saving and spending habits do not change? (Justify your answer by setting up the terms in the sequence that are to be added indefinitely, *i.e.*, show the total amount of spending in terms of a series, and find its sum.)

   Solution: The initial amount spent is \$50 billion. Let $c_1 = 50$.

   The original recipients of the rebate spend 0.92(50) billion dollars. Let $c_2 = 0.92(50)$.

   This amount c_2 becomes new income, so that 0.92(0.92(50)) is spent to yield $c_3 = 50(0.92)^2$.

   If this process repeats indefinitely, then the total amount spent is

$$T = 50 + 50(0.92) + 50(0.92)^2 + \ldots = \sum_{n=0}^{\infty} 50(0.92)^n,$$

   which is a convergent geometric series with $a = 50$ and $r = 0.92$, so that $T = \frac{50}{1-0.92} = 625$.

   Therefore, the total amount of spending generated is \$625 billion (including the initial spending by the government).