MATH 136 SOLUTIONS for Assignment 6 Fall/05

Was Due: Thursday Nov. 3/05
(Grade is out of 50.)
ONLY the following problems are marked:! The others are just checked to see an attempt is made.
1,3,4,5,7,9

1 page 127 #34

4 marks

Let

$$D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}.$$

Then the LU factorization for $A = LU$ can be found from page 127 # 33. We have:

$$A = LU = \bar{A}D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix}$$

Therefore, $A^{-1} = U^{-1}L^{-1}$, where from page 127 # 33. We have:

$$\bar{L}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{pmatrix}.$$

So

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{n} \end{pmatrix}$$

2 page 133 #16

4 marks

Solution 1:
Denote the 5×5 matrix by A. then $\forall y \in R^5$, since A is invertible, $\exists x = A^{-1}y$. So $Ax = A(A^{-1}y) = Iy = y$. So $R^5 \subseteq \text{col}(A) = \text{span}\{\text{columns of } A\}$. This yields the desired result.

Solution 2:
A is invertible, by *The invertible Matrix Theorem 8h*, the five column vectors of A span R^5.

3 page 133 #28

4 marks

Note!! This is only true if B is square, since invertibility does make sense otherwise.

Since there exists an inverse for AB, we can define $C = (AB)^{-1}A$. Therefore, $CB = ((AB)^{-1}A)B = (AB)^{-1}(AB) = I$. By Theorem 8 j, B is invertible.

4 page 139 #14

4 marks

If A_{11}^{-1} and A_{22}^{-1} are invertible, then there exists a square matrix:

$$B = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{pmatrix}.$$

We can then verify that $BA = I$, so A is invertible by Theorem 8 j.

Suppse A is invertible, and A has been partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11}, A_{22} are, respectively, $p \times p, q \times q$ matrices; and there is an inverse matrix B which we partition as: $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B_{11}, B_{22} are respectively p*p, q*q matrices. So $AB = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$. Then we have:

$$A_{11}B_{11} + A_{12}B_{21} = I_p A_{11}B_{12} + A_{12}B_{22} = 0 A_{22}B_{21} = 0 A_{22}B_{22} = I_q$$

Because A_{22}, B_{22} are q*q square matrices, so A_{22} is invertible by theorem 8k. And because A_{22} is invertible, from the third equation, we conclude $B_{21} = 0$. So the first equation is: $A_{11}B_{11} = I_p$. Using Theorem 8 k again, we notice A_{22}, B_{22} are both square and we conclude A_{11} is invertible.

5 page 140 #16

4 marks

Denote

$$P = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, Q = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then:

$$A = P \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} Q$$

Since A is invertible, by the conclusion of page 139 #14, P and Q are both invertible. Thus, we can multiply on the left by the inverse of P and on the right by the inverse of Q and obtain

$$W = P^{-1}AQ^{-1} = \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}.$$

Now W is invertible, i.e.

$$W^{-1} = QA^{-1}P = \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}^{-1}.$$

The fact that S is invertible now follows from page 139 #14 proved above.

6 page 149 #2, #12

4 marks

#2

$$\text{solve } Ax = LUx = b; \text{ or solve } Ly = b \text{ and then } Ux = y$$

First we solve $Ly = b$ and get $y = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$; then solve $Ux = y$ to get $x = \begin{pmatrix} \frac{1}{4} \\ 2 \\ 1 \end{pmatrix}$

If we use row reduction, the MATLAB function is:

```
A=[ 4 3 -5 2; -4 -5 7 -4; 8 6 -8 6]; A(1,:)=A(1,:)/A(1,1);
A(2,:)=A(2,:)-A(1,:)*A(2,1); A(3,:)=A(3,:)-A(3,1)*A(1,:);
A(2,:)=A(2,:)/A(2,2);A(3,:)=A(3,:)-A(3,2)*A(2,:);
A(3,:)=A(3,:)/A(3,3); A(1,:)=A(1,:)-A(1,3)*A(3,:);
A(2,:)=A(2,:)-A(2,3)*A(3,:);A(1,:)=A(1,:)-A(1,2)*A(2,:)
```

then the output is as follows:

```
A =

    1.0000    0.7500   -1.2500    0.5000
         0    1.0000   -1.0000    1.0000
         0         0    1.0000    1.0000


A =

    1.0000    0.7500   -1.2500    0.5000
         0    1.0000   -1.0000    1.0000
         0         0    1.0000    1.0000


A =

    1.0000         0         0    0.2500
         0    1.0000         0    2.0000
         0         0    1.0000    1.0000
```

thus the result is

$$\begin{pmatrix} 0.25 \\ 2 \\ 1 \end{pmatrix}$$

One can also use the LU command in MATLAB (though the L is permuted)

```
!rm output.txt
diary output.txt
clear all
A=[
4 3 -5
-4 -5 7
8 6 -8]
b=[2;-4;6]
[L,U]=lu(A)
y=L\b
x=U\y
```

```
A =

     4     3    -5
    -4    -5     7
     8     6    -8


b =

     2
    -4
     6


L =

    0.5000         0    1.0000
   -0.5000    1.0000         0
    1.0000         0         0


U =

     8     6    -8
     0    -2     3
     0     0    -1


y =

     6
    -1
    -1


x =

    0.2500
    2.0000
    1.0000
```

#12 The MATLAB program for the LU factorization is:

```
!rm output.txt
diary output.txt
clear all
A=[2 -4 2; 1 5 -4;-6 -2 4];
A0=A;
I1=eye(3);
I2=eye(3);
I3=eye(3);
I1(2,:)=I1(2,:)-I1(1,:)*A0(2,1)/A0(1,1)
A=I1*A0
I2(3,:)=I2(3,:)-I2(1,:)*A(3,1)/A(1,1)
A=I2*I1*A0
I3(3,:)=I3(3,:)-I3(2,:)*A(3,2)/A(2,2)
L=I3*I2*I1
U=L*A0
diary off
```

the output follows:

```
I1 =

    1.0000         0         0
   -0.5000    1.0000         0
         0         0    1.0000


A =

     2    -4     2
     0     7    -5
    -6    -2     4


I2 =

     1     0     0
     0     1     0
     3     0     1


A =

     2    -4     2
     0     7    -5
```

```
     0   -14    10


I3 =

     1     0     0
     0     1     0
     0     2     1


L =

    1.0000         0         0
   -0.5000    1.0000         0
    2.0000    2.0000    1.0000


U =

     2    -4     2
     0     7    -5
     0     0     0
```

7 page 223 #6, #8

6 marks

#6 Denote the set $S := \{p(t)|p(t) = a + t^2\}$. S is NOT a subspace, since multiplication by the scalar 2 yields $q(t) = 2a + 2t^2$ which is not in S.

#8 Denote $S := \{p(t)|p(0) = 0\}$. Then:

$$\forall p_1, p_2 \in S, p_1 + p_2 \in P_n, \text{ and } (p_1 + p_2)(0) = p_1(0) + p_2(0) = 0.$$

So $p_1 + p_2 \in S$. And we get closure by scalar multiplication

$$\forall c \in R, \forall p \in S, cp(0) = c0 = (cp)(0).$$

Thus S is a subspace.

8 page 234 #5, #16

8 marks

#5 We write out the general solution for Ax=0: here x_2, x_4 are free variables, and

$$Nul(A) = \{x|Ax=0\} = \{c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{pmatrix} |c_1, c_2 \in R\}$$

#16

$$\left\{ \begin{pmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{pmatrix} |b,c,d \in R \right\} = \left\{ \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} | \begin{pmatrix} b \\ c \\ d \end{pmatrix} \in R \right\}$$

if take $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{pmatrix}$, then

$$\left\{ \begin{pmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{pmatrix} |b,c,d \in R \right\} = \left\{ Ax|x \in R^3 \right\} = \mathrm{col}\,(A)$$

9 page 235 #28, #32

8 marks

28 Since the coefficients matrices are same, and b in the second system is 5 times of the first system, so we could denote the two systems as:

$$Ax = b, Ax = 5b$$

then if $\exists x$ such that $Ax = b$, then we have $A(5x) = 5(Ax) = 5b$. So $5x$ is the solution to the second system.

32

$$T(p) = \begin{pmatrix} a_0 \\ a_0 \end{pmatrix}, \quad \forall p(t) = a_2t^2 + a_1t + a_0.$$

Therefore, the

$$\text{kernel}\,(T) = \{p(t) = a_2t^2 + a_1t + a_0 : a_0 = 0\}.$$

Define

$$p_1(t) = t, \quad p_2(t) = t^2.$$

Then both are in kernel(T). And moreover, $p(t) = a_2t^2 + a_1t = a_2p_2(t) + a_1p_1(t)$, i.e. the two polynomials span the kernel.

10 page 236 #36

4 marks

We are given that T is a linear transformation from V to W and Z is a subspace of W. And

$$U := \{x | T(x) \in Z, x \in V\}.$$

To show that U is a subspace, we ONLY need to show closure under summation and scalar multiplication. Equivalently, we need only show that

$$u_1 + cu_2 \in U, \quad \forall u_1, u_2 \in U, \forall \text{scalars } c \tag{1}$$

But

$$T(u_1 + cu_2) = T(u_1) + cT(u_2) \in Z \quad \forall u_1, u_2 \in U, \forall \text{scalars } c$$

since Z is a subspace and T is a linear transformation. Therefore (1) follows.