MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 9

1: (a) Approximate $\sqrt{110}$ so that the error is smaller than $\frac{1}{1000}$.

Solution: Let $f(x)=\sqrt{100+x}=10\big(1+\frac{x}{100}\big)^{1/2}=10\big(1+\frac{1}{2}\,\frac{x}{100}+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\big(\frac{x}{100}\big)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\big(\frac{x}{100}\big)^3+\cdots\big)=10+\frac{1}{2\cdot 10}x-\frac{1}{2^2 2!10^3}x^2+\frac{1\cdot 3}{2^3 3!10^5}x^3-\frac{1\cdot 3\cdot 5}{2^4 4!10^7}x^4+\cdots$. Then $\sqrt{110}=f(10)=10+\frac{1}{2}-\frac{1}{2^2 2!10}+\frac{1\cdot 3}{2^3 3!10^2}-\frac{1\cdot 3\cdot 5}{2^4 4!10^3}+\cdots\cong 10+\frac{1}{2}-\frac{1}{80}=\frac{839}{80}$ with absolute error $E<\frac{1\cdot 3}{2^3 3!10^2}=\frac{1}{1600}<\frac{1}{1000}$ by the A.S.T.

(b) Approximate $\ln(5/6)$ so that the error is smaller than $\frac{1}{2000}$.

Solution: We provide two solutions. First, let $f(x)=\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots$. Then $\ln\big(\frac{5}{6}\big)=f\big(\frac{1}{6}\big)=-\frac{1}{6}-\frac{1}{2\cdot 6^2}-\frac{1}{3\cdot 6^3}-\frac{1}{4\cdot 6^4}-\cdots\cong-\frac{1}{6}-\frac{1}{2\cdot 6^2}-\frac{1}{3\cdot 6^3}=-\frac{1}{6}-\frac{1}{72}-\frac{1}{648}=-\frac{118}{648}=-\frac{59}{324}$ with absolute error $E=\frac{1}{4\cdot 6^4}+\frac{1}{5\cdot 6^5}+\frac{1}{6\cdot 6^6}+\cdots<\frac{1}{4\cdot 6^4}+\frac{1}{4\cdot 6^5}+\frac{1}{46^6}+\cdots=\frac{\frac{1}{4\cdot 6^4}}{1-\frac{1}{6}}=\frac{1}{20\cdot 6^3}=\frac{1}{4320}<\frac{1}{2000}$, by the C.T.

For the second solution, we let $g(x)=\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$. Then we have $\ln\big(\frac{5}{6}\big)=-\ln\big(\frac{6}{5}\big)=-g\big(\frac{1}{5}\big)=-\frac{1}{5}+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}+\frac{1}{4\cdot 5^4}-\cdots\cong-\frac{1}{5}+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}=-\frac{1}{5}+\frac{1}{50}-\frac{1}{375}=-\frac{137}{750}$, with absolute error $E\le\frac{1}{4\cdot 5^4}=\frac{1}{2500}<\frac{1}{2000}$ by the A.S.T.

2: Approximate $\displaystyle\int_0^{1/5}\frac{\ln(1+x)}{x}\,dx$ so that the error is less than $\frac{1}{1000}$.

Solution: $\displaystyle\int_0^{1/5}\frac{\ln(1+x)}{x}\,dx=\int_0^{1/5}\frac{x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots}{x}\,dx=\int_0^{1/5}1-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3+\cdots\,dx=$ $\Big[x-\frac{1}{2^2}x^2+\frac{1}{3^2}x^3-\frac{1}{4^2}x^4+\cdots\Big]_0^{1/5}=\frac{1}{5}-\frac{1}{2^2 5^2}+\frac{1}{3^2 5^3}+\frac{1}{4^2 5^4}+\cdots\cong\frac{1}{5}-\frac{1}{2^2 5^2}=\frac{19}{100}$ with absolute error $E<\frac{1}{3^2 5^3}=\frac{1}{1125}<\frac{1}{1000}$, by the A.S.T.

3: (a) Evaluate the limit $\displaystyle\lim_{x\to 0}\frac{\sin x\tan^{-1}x-x^2}{(1-\cos x)^2}$.

Solution: We have $\displaystyle\lim_{x\to 0}\frac{\sin x\tan^{-1}x-x^2}{(1-\cos x)^2}=\lim_{x\to 0}\frac{\big(x-\frac{1}{6}x^3+\cdots\big)\big(x-\frac{1}{3}x^3+\cdots\big)-x^2}{\big(1-\big(1-\frac{1}{2}\,x+\cdots\big)\big)^2}=\lim_{x\to 0}\frac{-\frac{1}{2}x^4+\cdots}{\big(\frac{1}{2}x^2+\cdots\big)^2}$ $\displaystyle=\lim_{x\to 0}\frac{-\frac{1}{2}x^4+\cdots}{\frac{1}{4}x^4+\cdots}=-2.$

(b) Find the ninth derivative $f^{(9)}(0)$, where $f(x)=(8+x^3)^{2/3}$.

Solution: $f(x)=(8+x^3)^{2/3}=4\big(1+\frac{x^3}{8}\big)^{2/3}=4\Big(1+\frac{2}{3}\,\frac{x^3}{8}+\frac{\big(\frac{2}{3}\big)\big(-\frac{1}{3}\big)}{2!}\big(\frac{x^3}{8}\big)^2+\frac{\big(\frac{2}{3}\big)\big(-\frac{1}{3}\big)\big(-\frac{4}{3}\big)}{3!}\big(\frac{x^3}{8}\big)^3+\cdots\Big)$, so $c_9=\frac{4\cdot 2\cdot 1\cdot 4}{3^3\cdot 3!\cdot 8^3}=\frac{1}{3^4\,2^5}$ and $f^{(9)}(0)=9!\,c_9=\frac{9!}{3^4\,2^5}=140$.

4: Find the Taylor polynomial of degree 5 centered at 0 for the solution $y=f(x)$ to the initial-value problem $(1-x)\,y''-2y=4$ with $y(0)=1$ and $y'(0)=3$.

Solution: Let $y=a_0+a_1x+\cdots+a_5x^5+\cdots$ be the solution. Since $y(0)=1$ we have $a_0=1$ and since $y'(0)=3$ we have $a_1=3$, so $y=1+3x+a_2x^2+\cdots a_5x^5+\cdots$, $y'=3+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots$ and $y''=2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots$, so

$$\begin{aligned}0&=(1-x)y''-2y-4\\&=(1-x)\big(2a_2+6a_3x+12a_4x^2+20a_5x^3+\cdots\big)-2\big(1+3x+a_2x^2+a_3x^3+\cdots\big)-4\\&=(2a_2-2-4)+(6a_3-2a_2-6)x+(12a_4-6a_3-2a_2)x^2+(20a_5-12a_4-2a_3)x^3+\cdots.\end{aligned}$$

$(2a_2-6)=0\Longrightarrow a_2=3$, $(6a_3-2a_2-6)=0\Longrightarrow a_3=2$, $(12a_4-6a_3-2a_2)=0\Longrightarrow a_4=\frac{18}{12}=\frac{3}{2}$ and $(20a_5-12a_4-2a_3)=0\Longrightarrow a_5=\frac{22}{20}=\frac{11}{10}$. Thus $y=f(x)=1+3x+3x^2+2x^3+\frac{3}{2}x^4+\frac{11}{10}x^5+\cdots$, and so the Taylor polynomial of degree 5 for $f(x)$ is $T_5(x)=1+3x+3x^2+2x^3+\frac{3}{2}x^4+\frac{11}{10}x^5$.

5: Evaluate the sum of each of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!}$

Solution: Recall that $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}$, so $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, (\sqrt{2})^{2n} = \cos(\sqrt{2})$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)!}$

Solution: We have $e^x - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ so $\displaystyle\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$. Differentiate both sides to get $\displaystyle\frac{x\, e^x - (e^x - 1)}{x^2} = \sum_{n=0}^{\infty} \frac{n\, x^{n-1}}{(n+1)!}$. Then put in $x = 1$ to get $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n+1)!} = 1$.