Math 137 Physics Based Section Assignment 5

(Q1) The air temperature, T, in a railway tunnel is given by a function of the distance, x, measured in meters from the entrance. We assume

$$T(x) = x(1000 - x) + 20.$$

i) If the tunnel is 1 kilometer long sketch $T(x)$.
ii) If the train's position is given by $x(t) = t/3$ where time is measured in seconds, find the rate of change of temperature with time measured by a thermometer mounted on the train.
iii) Repeat part ii) for

$$x(t) = \frac{t(1000 - t)}{1000}$$

where $0 \leq t \leq 1000$ and comment on the path taken by the train.
iv) Now consider a situation in which the velocity of the train is specified as

$$v(t) = \frac{t}{3}.$$

If $x(0) = 0$ find the rate of change of temperature with time.
v) Can you do the problem if you do not know $x(0)$? Why or why not?
vi) If $T(x)$ is an increasing function of x can the measured rate of change be negative? explain.
vii) Give all possible conditions on $T(x)$ and $x(t)$ so that the rate of change with time measured is positive.

(Q2)i) Differentiate $a(t) = \sin(\cos(t))$ with respect to time.
ii) Consider sine and its inverse and write, from the definition of inverse,

$$\sin^{-1}(\sin(t)) = 1.$$

Let $y = \sin t$ and use the chain rule to apply the derivative machine with respect to t of the above equation.
iii) Use

$$\sin^2 t + \cos^2 t = 1$$

to derive the identity

$$\frac{d}{dy}\left[\sin^{-1} y\right] = \frac{1}{\sqrt{1 - y^2}}.$$

iv) Use the same kind of trick to derive the identity

$$\frac{d}{dy}\left[\tan^{-1} y\right] = \frac{1}{1 + y^2}$$

(Q3) If y is a function of t and

$$y^3 + ty - \frac{t^5}{6} = 1$$

apply the derivative machine with respect to t to find $y'(1)$ given that $y(1) = 2$.

(Q4) Use change of variables to find $x(t)$ given
i) $v(t) = t\cos(t^2)$
ii)

$$v(t) = \frac{3t^3 + 6t}{t^4 + 4t^2}$$

iii) $v(t) = \sin^3(t)\cos(t)$
iv) $v(t) = \cos^2(2t)\sin t \cos t$

(Q5) If $x(t) = \sin\sqrt{t}$ show that $v(t) = x'(t)$ tends to zero as $t \longrightarrow \infty$ and find T so that $t > T$ guarantees $|v(t)| < 0.001$

(Q6)i) Show that both $x(t) = \sin(\omega t)$ and $x(t) = \cos(\omega t)$ satisfy the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which is called the **SIMPLE HARMONIC OSCILLATOR**. Can you guess why the differential equation is called what it is?
ii) Show that for any two numbers A and B

$$x(t) = A\sin\omega t + B\cos\omega t$$

is also a solution of the same differential equation.
iii) Find A and B so that $x(0) = 0$ and $v(0) = 5$.
iv) Find A and B so that $x(0) = 5$ and $v(0) = 0$.
v) The general solution can also be written as

$$x(t) = A_0\cos(\omega t + \delta)$$

where A_0 is called the amplitude and δ is called the phase shift. Find A_0 and δ corresponding to parts iii) and iv) (you will need to use the addition formula for cosine).