MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 10 SOLUTIONS

NOT TO BE HANDED IN

1. Find and sketch the domain of the function $f(x,y) = \ln(9 - x^2 - 9y^2)$.

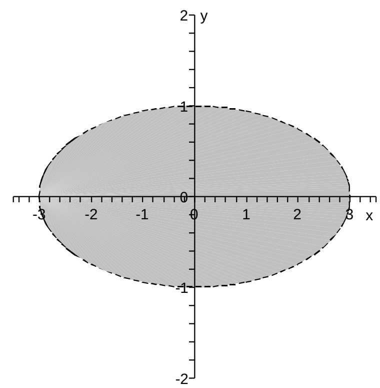
Solution: The only restriction on f is that the argument of ln must be greater than zero:

$$9 - x^2 - 9y^2 > 0 \Rightarrow 9 > x^2 + 9y^2 \Rightarrow 1 > \left(\frac{x}{3}\right)^2 + y^2$$

which is the region inside an ellipse which is centered at $(x,y) = (0,0)$ and has a semi-major axis of 3 and semi-minor axis of 1.
Thus, the domain of f is

$$D = \left\{(x,y) \,|\, 1 > \frac{x^2}{9} + y^2\right\},$$

(where the original inequality can also be used instead of the simplified one).



In more detail (to help with the sketch and solving inequalities and simplifying square roots, if necessary):

(i) Set $y = 0$ in the inequality to yield

$$9 > x^2 \Rightarrow \sqrt{9} > \sqrt{x^2} \Rightarrow \begin{cases} 3 > x, & \text{if } x > 0 \\ 3 > -x, & \text{if } x < 0. \end{cases}$$

When $x < 0$, we can rewrite the inequality by dividing both sides by -1 to obtain $x > -3$, ensuring we 'flipped' the inequality. To justify that 'flip', here is the same result obtained by using addition/subtraction instead:

$$3 - 3 > -x - 3 \Rightarrow 0 > -x - 3 \Rightarrow 0 + x > -x - 3 + x \Rightarrow x > -3.$$

(ii) Likewise, set $x = 0$ in the inequality to yield

$$9 > 9y^2 \Rightarrow 1 > y^2 \Rightarrow \sqrt{1} > \sqrt{y^2} \Rightarrow 1 > y \text{ if } y > 0 \text{ and } 1 > -y \text{ if } y < 0 \text{ or } -1 < y.$$

2. Draw a contour map of the surface $f(x, y) = (y - 2x)^2$. Describe the surface.

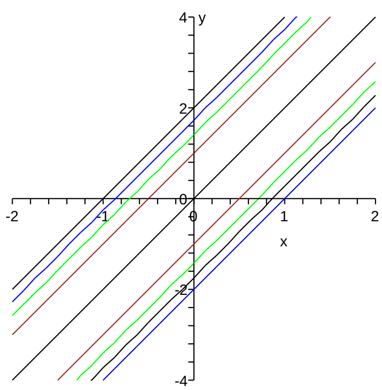
   Solution: Contours of $z = f(x, y)$ are obtained by letting $z = k$, where k is constant, and plotting the results in the xy−plane. (Note that, since $z \geq 0$, that the constant $k \geq 0$.)

   For example, if $z = k = 0$, then $(y - 2x)^2 = 0 \Rightarrow \sqrt{(y - 2x)^2} = \pm\sqrt{0} \Rightarrow y - 2x = 0 \Rightarrow y = 2x$, a straight line through the origin of slope 2.

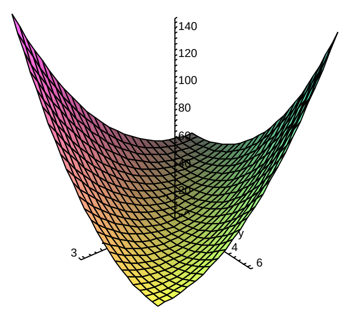
   If $z = k = 1$, then $(y - 2x)^2 = 1 \Rightarrow y - 2x = \pm 1 \Rightarrow y = 2x \pm 1$, two different straight lines, both of slope 2, where one intersects the y−axis at $y = -1$, the other at $y = 1$. (Note that these two lines are parallel to each other and to the one for $k = 0$. These two lines are equidistant from the line for $k = 0$.)

   In general, if $z = k \geq 0$, then $(y - 2x)^2 = k \Rightarrow y - 2x = \pm\sqrt{k} \Rightarrow y = 2x \pm \sqrt{k}$, a set of parallel straight lines of slope 2 that intersect the y−axis at $\pm\sqrt{k}$. Each pair of lines for a value of k are equidistant from the line for $k = 0$.

   Below are **contours** for $k = 0, 1, 2, 3, 4$.

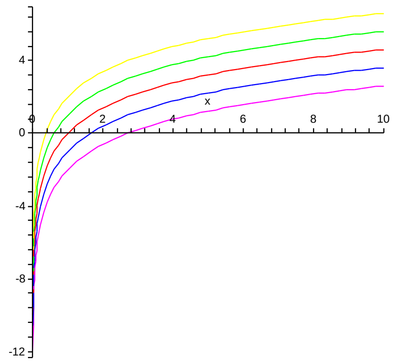
   Given that the contours are parallel lines which progressively become closer to each other as z increases (seen with increasing k), the **surface** has a symmetry along the contour line $y = 2x$. The surface can be described as a smooth plane whose opposite corners are pulled upward (somewhat like a parabolic cylinder could be considered a smooth plane that has been symmetrically been pulled upward at its edges). The surface is provided below for your reference.

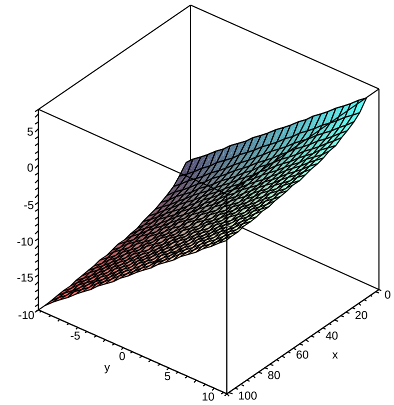
   

3. Draw a contour map of the surface $f(x, y) = y - 2\ln x$. How does the surface differ from that of a plane?

   Solution: Let $z = f(x, y) = k \Rightarrow y - 2\ln x = k \Rightarrow y = 2\ln x + k$, for constant values k. The contours are then simply the curve $y = 2\ln x$ shifted upward or downward a constant value (k). The contours below are for $k = 2, 1, 0, -1, -2$, where $k = 2$ is the top contour in the figure, $k = 1$ is the second from the top, and so on.

   The parallel nature of parts of the contours imply the surface resembles a plane for those x, y, and z values. Where the contours become closer to each other as z changes for decreasing x and negative y values is where the surface begins to curl, losing its planar nature.

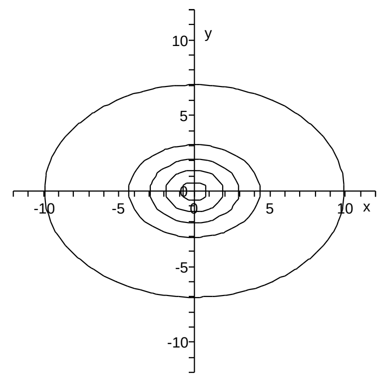




4. A thin metal plate, located in the $xy-$plane, has temperature $T(x, y)$ at the point (x, y). The level curves of T are called *isothermals* because at all points on an isothermal the temperature is the same.

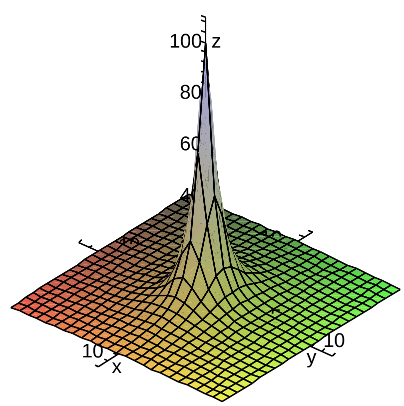
   (a) Sketch some isothermals if the temperature function is given by $T(x, y) = \dfrac{100}{1 + x^2 + 2y^2}$.

   Solution: Set $T = k$, for constant k to give $x^2 + 2y^2 = \dfrac{100}{k} - 1$, a set of ellipses centered at $(x, y) = (0, 0)$. Since $T > 0$ by its definition (*i.e.*, its equation demands $T > 0$ since all terms are greater than or equal to 0, yielding the range $T \in (0, 100]$), we select values $k > 0$. As $z = k$ increases, the contours get closer together as they approach the origin $(x, y) = (0, 0)$. Contours presented are for $T = 1, 5, 10, 20, 50$.

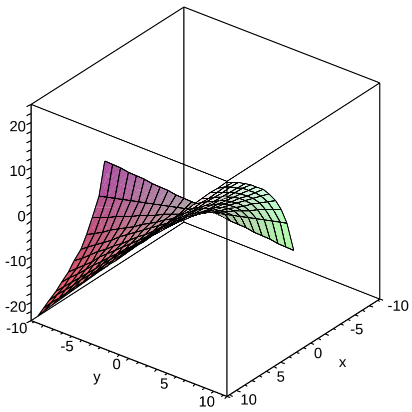


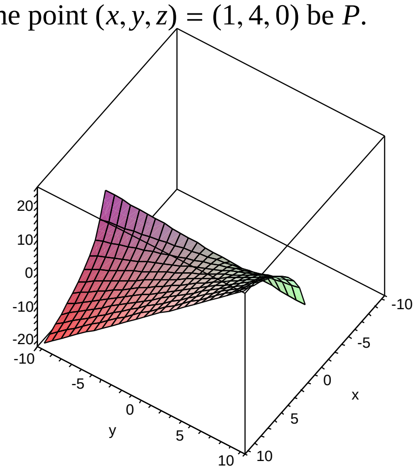
   (b) Sketch the function $T(x, y)$.

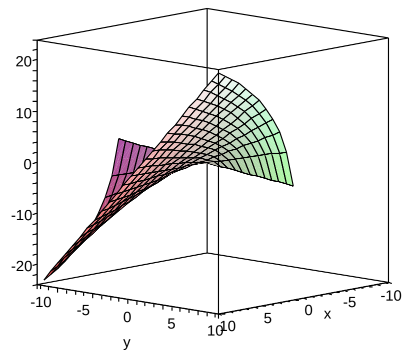
   Solution: The surface begins just above the $xy-$plane, since $T > 0$ for large x and y, and increases as x and y decrease. Given that the contours become closer together near the origin implies the surface increases greater near the origin (*i.e.*, the surface has greater slope near the origin).



5. Let $z = f(x, y) = y \ln x$ and the point $(x, y, z) = (1, 4, 0)$ be P.







(a) Find an equation of the tangent plane to $f(x, y)$ at P.

Solution: $z = f(x, y) = y \ln x \Rightarrow f_x(x, y) = \dfrac{y}{x}$, $f_y(x, y) = \ln x$, so that $f_x(1, 4) = 4$, $f_y(1, 4) = \ln 1 = 0$. Then the equation of the tangent plane is $z - 0 = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4)$ or $z = 4x - 4$. We can also express this equation of the plane in the form $Ax + By + Cz = D$: $4x - z = 4$.

(b) Find the gradient of f.

Solution: $\nabla f(x, y) = f_x(x, y)\vec{\imath} + f_y(x, y)\vec{\jmath} = \dfrac{y}{x}\vec{\imath} + \ln x\,\vec{\jmath} = \left\langle \dfrac{y}{x}, \ln x \right\rangle$.

(This is the general slope of f at an arbitrary point.)

(c) Evaluate the gradient of f at P.

Solution: $\nabla f(1, 4, 0) = \langle 4, 0 \rangle$.

(This is the general slope of f at the point P.)

(d) Find the rate of change of f at P in the direction $\vec{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$.

Solution: Since $\vec{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$ has length 1 it is already a unit vector (*i.e.*, since $\sqrt{(-4/5)^2 + (3/5)^2} = \sqrt{16/25 + 9/25} = \sqrt{25/25} = 1$, the vector has length of 1 *unit*). (If the length or magnitude of the vector was not 1, we would divide the given vector by its length to "normalize" it so that it has a length of 1 unit.)

$$D_{\vec{u}} f(1, 4, 0) = \nabla f(1, 4) \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = \langle 4, 0 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = 4\left(-\frac{4}{5}\right) + 0 \cdot \frac{3}{5} = -\frac{16}{5}.$$

(This is the slope of f at P in the specific direction $\vec{u}$.) Since the directional derivative of f at the point P is negative, then the surface f is sloping downward.

(e) What is the maximum rate of change of f at P?

Solution: The surface f changes the fastest in the direction of the gradient at P:

$|\nabla f(1, 4)| = \sqrt{4^2 + 0^2} = \sqrt{16} = 4$.

(This value is the largest slope of f at the point P. So, "looking in other directions" from this point P, the slope is less than 4. It is comparable to being on a hill and realizing that the slope seems less steep if you descend the hill on some sort of angle or diagonal, rather than following the steepest part. Likewise, cycling up a very steep hill is easier if you *don't* follow the gradient or steepest slope, and this is done by cycling in a zig-zag pattern uphill so that you experience the smallest possible slope.)

6. The radius of a right circular cone is increasing at a rate of 4.6 cm/s while its height is decreasing at a rate of 6.5 cm/s. At what rate is the volume of the cone changing when the radius is 300 cm and the height is 350 cm?

   Solution: The volume of the cone changes in time, so is given by $V(t) = \pi r(t)^2 h(t)/3\ cm^3$, where $\dfrac{dr}{dt} = 4.6\ cm/s$, $\dfrac{dh}{dt} = -6.5\ cm/s$. Thus,

   $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi}{3} rh\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}.$$

   When $r = 300, h = 350$, so:

   $$\frac{dV}{dt} = \frac{2\pi}{3}(300)(350)(4.6) + \frac{\pi(300)^2}{3}(-6.5) = 322,000\pi - 195,000\pi = -127,000\pi\ cm^3/s,$$

   thus, the volume decreases at a rate of $127,000\pi\ cm^3/s$ when $r = 300\ cm$ and $h = 350\ cm$.

7. Find the work done by the force field $\vec{F}(x, y, z) = z\vec{i} + y\vec{j} - x\vec{k}$ on a particle that moves along the path $\vec{r}(t) = t\vec{i} + \sin t\vec{j} + \cos t\vec{k}$, $0 \le t \le \pi$.

   Solution: $W = \displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$, where $\vec{F}(\vec{r}(t)) = \langle \cos t, \sin t, -t\rangle$, $t \in [0, \pi]$ and $\vec{r}\,'(t) = \langle 1, \cos t, -\sin t\rangle$:

   $$\begin{aligned} W &= \int_0^\pi \langle \cos t, \sin t, -t\rangle \cdot \langle 1, \cos t, -\sin t\rangle dt = \int_0^\pi (\cos t + \sin t\cos t + t\sin t)dt \\ &= \left[\sin t - \frac{\cos 2t}{4} - t\cos t + \sin t\right]_0^\pi = \pi \end{aligned}$$

   (where we have used $\sin t\cos t = 2\sin t$, whose antiderivative is $\frac{1}{2}\cos 2t$, and integration by parts to antidifferentiate $t\sin t$).