Math 138 physics based
Assignment #7

Q1 i) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ here $a_n = \frac{1}{n^2+1} \le \frac{1}{n^2}$

for all $n = 1, 2, 3, \dots$

but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

ii) Let $f(x) = \frac{1}{x^2+1}$ so that $f(n) = \frac{1}{n^2+1}$

$$f'(x) = \frac{-2x}{(x^2+1)^2} < 0 \quad \text{for } x \ge 1$$

so $f(\cdot)$ is decreasing on $[1, \infty)$

Next $\int_1^{\infty} f(x)\,dx = \lim_{L \to \infty} \int_1^{L} \frac{1}{x^2+1}\,dx = \lim_{L \to \infty} \arctan x \Big|_1^L$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ because $\lim_{L \to \infty} \arctan L = \frac{\pi}{2}$

$\arctan(1) = \frac{\pi}{4}$

$\therefore$ by the Integral Test

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Q1 iii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{4-1} + \frac{1}{9-1} + \frac{1}{16-1} + \cdots$

$$= \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots$$

$$\leq \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \cdots}_{=}$$

or $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$

converges provided $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$ does

but $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$

which converges by the p-test

iv) $a_n = \dfrac{1}{n^2-1} = \dfrac{1}{(n+1)(n-1)}$ by difference of squares

$$= \frac{-1/2}{n+1} + \frac{1/2}{n-1}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2+1} = \sum_{n=2}^{\infty} \left(-\frac{1}{2} \frac{1}{n+1} + \frac{1}{2} \frac{1}{n-1} \right)$$

where $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+1}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$ both diverge

Q1 iv) but the two divergent series have the opposite sign and hence could cancel.

Indeed splitting an unknown series into two divergent parts with a minus between is in general not helpful.

Q1v) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ the trouble here is that the n's subtracted so using comparison test will be tricky

The integral test with $f(x) = \frac{1}{x^2-x}$

will work but needs care because
$$\int_2^{\infty} f(x)\,dx = \lim_{L\to\infty}\left(\ln(x-1) - \ln(x)\right)\Big|_2^L$$
needs a bit of care:
Write
$$\ln(x-1) - \ln(x) = \ln\left(\frac{x-1}{x}\right) = \ln\left(1-\frac{1}{x}\right)$$
so $\displaystyle\int_2^{\infty} f(x)\,dx = -\ln\left(1-\frac{1}{2}\right)$

what about ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2-(n+1)}{n^2-n} = \frac{n^2 + \text{lower order}}{n^2 + \text{lower order}}$

and we get the dreaded $\left|\frac{a_{n+1}}{a_n}\right| \to 1$

vi) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n} = \sum_{n=1}^{\infty} (-1)^n p_n$

since $\ln(n)$ grows slower than n

$\frac{\ln n}{n} \to 0$ as $n \to \infty$

$p_{n+1} = \frac{\ln(n+1)}{n+1}$ is this $\le p_n$?

Consider $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{1}{x^2} - \frac{\ln(x)}{x^2}$$

This is not < 0 for all $x > 1$
but it is < 0 for all $x > e$

So let $\sum_{n=1}^{\infty} (-1)^n p_n = \sum_{n=1}^{4} (-1)^n p_n + \sum_{n=5}^{\infty} (-1)^n p_n$

The finite series doesn't matter
so concentrate on $\sum_{n=5}^{\infty} (-1)^n p_n$

From the above discussion & the alternating series test the series converges

Q1 vi) For absolute convergence we consider

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \quad \text{or} \quad a_n = \frac{\ln(n)}{n} > \frac{1}{n}$$

when $n > e$

so by comparison test with the divergent series

$$\sum_{n=5}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \quad \text{diverges}$$

& $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is <u>conditionally convergent</u>

Q1 vii) $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$

since n grows faster than $\ln(n)$

$(-1)^n \frac{n}{\ln(n)} \not\to 0$ so $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$ diverges

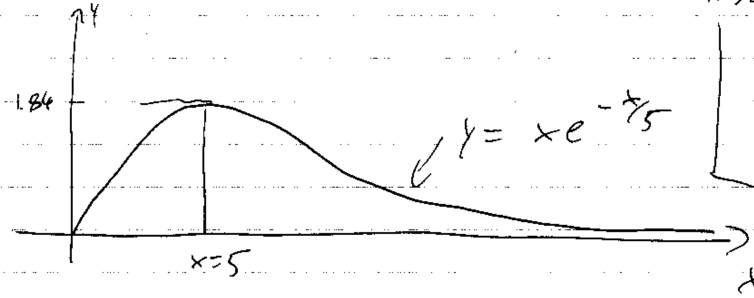
by Nth term test, and so does $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

Q2 i), ii), iii) $f(x) = x e^{-x/5}$

$$f'(x) = e^{-x/5} - \frac{x}{5} e^{-x/5}$$

So that $f'(x) = 0$ when $x = 5$

Moreover $f(0) = 0$ $f(5) = \frac{5}{e}$ $\lim_{x \to \infty} f(x) = 0$



hence $\lim_{n\to\infty} ne^{-n/5} = 0$

& Nth term test does not predict divergence

So based on this analysis write

$$\sum_{n=1}^{\infty} n \exp\{-n/5\} = \sum_{n=1}^{5} n \exp\{-n/5\}$$

$$+ \sum_{n=6}^{\infty} n \exp\{-n/5\}$$

and on $[6, \infty)$ $f(x)$ is decreasing

and $\int_6^x x \exp\{-x/5\} dx = \lim_{L\to\infty} -5(x+5) e^{-x/5}\Big|_0^L$

$$= 55 e^{-6/5}$$

Q2 ~~a~~ (cont'd) so that by the integral test $\sum_{n=6}^{\infty} n e^{-n/5}$ converges

and then so does $\sum_{n=1}^{\infty} n e^{-n/5}$

In summary, we split the series up so that the tail, or the part with an infinite number of terms, satisfies the hypotheses of the integral test.

Q3 $\sum_{n=1}^{\infty} e^{-n^2}$ Try comparison test & integral test together

$$a_n = e^{-n^2} \leq e^{-n} \quad \text{for } n = 1, 2, 3, \ldots$$

For $\sum_{n=1}^{\infty} e^{-n}$ let $f(x) = e^{-x}$; $f'(x) = -e^{-x} < 0$

$$\int_1^{\infty} e^{-x} dx = \lim_{L \to \infty} -e^{-x} \Big|_1^L = \frac{1}{e}$$

Q3 cont'd] ∴ by Integral test

$$\sum_{n=1}^{\infty} e^{-n} \quad \text{converges}$$

Finally by comparison test

$$\sum_{n=1}^{\infty} e^{-n^2} \quad \text{converges.}$$

Note: One could do the integral test first & then do the comparison theorem for integrals

Q4 i) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots$

ii) when $|x|<1$ we have

$$\left|(-1)^n \frac{x^n}{n!}\right| = \left|(-1)^n \frac{1}{n!} (\alpha)^n\right| \leq \alpha^n$$

where $\alpha = |x| < 1$

By comparison with the geometric series $\sum_{n=0}^{\infty} (\alpha)^n$

We get that $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ converges

absolutely & hence converges.

iii) Now x is general

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \to 0$$

as $n \to \infty$ for any fixed x

$\therefore \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ converges by ratio test

Q4 iv) differentiating term by term

$$\sum_{n=1}^{\infty} (-1)^n n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n-1}}{(n-1)!}$$

$$= -\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{(n-1)!} \qquad \text{let } k = n-1$$

$$= -\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

the negative of the original.

v) see attached plots

Q5 i) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{360} + \frac{x^8}{20160} - \ldots$$

$$= 1 + 0x - \frac{x^2}{2} + 0x^3 + \frac{x^4}{24} + \ldots$$

ii) If $|x| = \alpha$ then comparing with $\sum_{n=0}^{\infty} (\alpha^2)^n$ will yield absolute convergence

(Q5) iii) For a general x could argue using the alternating series test

The catch is that the series must be split up so that $p_{n+1} \le p_n$

Alternatively use the ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+2}}{(2n+2)!} \, \frac{(2n)!}{x^{2n}} \right|$$

$$= \frac{x^2}{(2n+2)(2n+1)} \to 0$$

as $n \to \infty$ for **any** fixed x

$\therefore$ by the ratio test $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges for all x

iv) $$p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$$

$$p'(x) = \sum_{n=1}^{\infty} (-1)^n \, 2n \, \frac{x^{2n-1}}{(2n)!} = -x + \frac{x^3}{3!} - \ldots$$

$$p''(x) = \sum_{n=\,}^{\infty} (-1)^n \, 2n(2n-1) \, \frac{x^{2n-2}}{(2n)!} = \sum_{n=2} (-1)^n \frac{x^{2n-2}}{(2n-2)!}$$

$$= (-1) \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2k!} = -p(x)$$

(Q6) i) $S_{N+1} = S_N + a_{N+1}$

$\therefore \quad a_{N+1} = S_{N+1} - S_N$

ii) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

so $a_n = b_n = \frac{1}{n}$ gives

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ diverges while

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} a_n b_n$ converges.

iii) Let $a_n = 1$ for all n
$b_n = -1$ for all n

then $a_n + b_n = 1 - 1 = 0$ for all n

$\therefore \quad \sum_{n=0}^{\infty} a_n + b_n = 0$

iv) Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

and $\sum_{n=1}^{\infty} n^2$ diverges

$\sum_{n=1}^{\infty} \frac{1}{n^2} n^2 = \sum_{n=1}^{\infty} 1$ and this series diverges

v) Let $a_n = \frac{1}{n^3}$ so that $\sum_{n=1}^{\infty} a_n$

converges by the p-test

and let $b_n = 1$ then $\sum_{n=1}^{\infty} b_n$ diverges

& $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

vi) I missed part vi)

vii) Let $a_n = \frac{1}{n^2}$ then $\sum_{n=1}^{\infty} a_n$ converges

and let $b_n = 1$ then

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \underbrace{\left(1 + \frac{1}{n^2}\right)}_{c_n}$$

and $c_n \to 1$ as $n \to \infty$

so get divergence by the N^{th} term test.

viii) No, this just property 4, pg. 96
- prove it ~~for~~ bonus if you wish.

(Q7) $\frac{dy}{dx} = y$; $y(0) = 5$

Try $p(x) = \sum_{n=0}^{\infty} a_n x^n$ $\quad p(0) = 5 \Rightarrow a_0 = 5$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\frac{dp}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$\frac{dp}{dx} = p(x)$ thus implies

$a_1 = a_0$; $2a_2 = a_1$ or $a_2 = \frac{a_1}{2}$

$a_3 = \frac{a_2}{3}$

$\vdots$

$a_n = \frac{a_{n-1}}{a_n}$

But $a_0 = 5$ $\therefore$

$$p(x) = 5 + 5x + 5\frac{x^2}{2} + 5\frac{x^3}{3\cdot 2} + 5\frac{x^4}{4\cdot 3\cdot 2} + \dots$$

Or $p(x) = 5 \sum_{n=0}^{\infty} a_n x^n$

(Q8) Just like Q7 but now get

$a_0 = 1$ from $y(0) = 1$ and $\frac{dp}{dx} = -p(x)$

implies

$a_1 = -a_0$, $2a_2 = -a_1$, ... $a_n = -\frac{a_{n-1}}{n}$

$$\therefore p(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3\cdot 2} - \dots$$

(Q8) cont'd $\quad p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

and from our past work we also know $\frac{dy}{dx} = -y \; ; \; y(0) = 1 \Rightarrow y(x) = e^{-x}$

$\therefore \; p(x) = e^{-x}$

Q9 i) $\quad$ Let $y = \sum_{n=0}^{\infty} a_n x^n \qquad y(0)=1 \Rightarrow a_0 = 1$

from the ODE
$$x \frac{dy}{dx} + y = 0$$

$$x\left[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots\right]$$
$$+ \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right] = 0$$

or $\quad a_0 + 2a_1 x + (2a_2 + a_2)x^2 + \dots = 0$

This holds for all $x > 0$ $\therefore$ $a_0 = a_1 = a_2 = \dots = 0$

If all coefficients are zero then $a_0 = 1$ is not possible & hence no solution exists.

ii) $\quad \frac{dy}{dx} - \frac{3y}{x} = 0 \qquad$ Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$y(0) = 0 \Rightarrow a_0 = 0 \qquad \frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

DE $\Rightarrow$ $(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots) - (3a_1 + 3a_2 x + 3a_3 x^2 + 3a_4 x^3 + \dots) = 0$

Q9 ii) cont'd

$$-2a_1 - a_2 x - 0x^2 + a_4 x^3 + \ldots = 0$$

This holds for all x only if

$$a_1 = a_2 = a_4 = a_5 = a_6 = \ldots = 0$$

BUT notice a_3 is undetermined

$\therefore$ $y(x) = a_3 x^3$ is a solution

check $y(0) = 0$ ✓

$$\frac{dy}{dx} = 3a_3 x^2 \qquad 3\frac{y}{x} = 3a_3 x^2$$

$$\therefore \frac{dy}{dx} - 3\frac{y}{x} = 0 \quad ✓$$

So the series solution gives a polynomial solution

NOTE: These problems were cooked to make a point, however series solutions of ODEs are an important theoretical and practical tool.