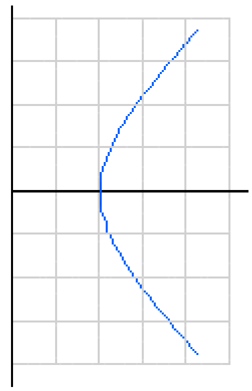
MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 5

1: Consider the parametric curve given by $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$.

(a) Sketch the curve.

Solution: We make a table of values then plot the curve.

t	x	y
4	17/4	15/4
3	10/3	8/3
2	5/2	3/2
1	2	0
1/2	5/2	−3/2
1/3	10/3	−8/3
1/4	17/4	−15/4



(b) Eliminate the parameter t to find an implicit equation for the curve.

Solution: We have $x^2 = t^2 + 2 + \frac{1}{t^2}$ and $y^2 = t^2 - 2 + \frac{1}{t^2}$, and so $x^2 - y^2 = 4$. This is an implicit equation for the curve. We also remark that since the curve only follows the right-hand branch of the hyperbola $x^2 - y^2 = 4$, it is given by the equation $x = \sqrt{4 + y^2}$.

(c) Find the equation of the tangent line to the curve at the point where $t = 2$.

Solution: We have $x' = 1 - \frac{1}{t^2}$ and $y' = 1 + \frac{1}{t^2}$. When $t = 2$, we have $(x, y) = \left(\frac{5}{2}, \frac{3}{2}\right)$ and $(x', y') = \left(\frac{3}{4}, \frac{5}{4}\right)$ so that $\frac{dy}{dx} = \frac{y'}{x'} = \frac{5}{3}$, and so the equation of the tangent line is $\left(y - \frac{3}{2}\right) = \frac{5}{3}\left(x - \frac{5}{2}\right)$, which can also be written as $5x - 3y = 8$.

(d) Find $\dfrac{d^2y}{dx^2}$ at the point where $t = 2$.

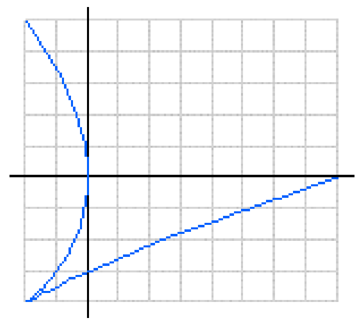
Solution: $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \dfrac{t^2 + 1}{t^2 - 1}$ so $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right)}{x'(t)} = \dfrac{2t(t^2-1) - 2t(t^2+1)}{(t^2-1)^2\left(1 - \frac{1}{t^2}\right)} = \dfrac{-4t^3}{(t^2-1)^3}$, and when $t = 2$ we have $\left.\dfrac{d^2y}{dx^2}\right|_{t=2} = -\frac{32}{27}$.

2: Consider the parametric curve given by $x = \frac{1}{2}(t^3 - 3t^2)$ and $y = t^2 - 4t$ with $-1 \le t \le 4$.

(a) Sketch the curve, showing all x and y-intercepts and all horizontal and vertical points.

Solution: We have $x = 0 \iff \frac{1}{2}(t^3 - 3t^2) = 0 \iff \frac{1}{2}\,t^2(t-3) = 0 \iff t = 0$ or 3, and we have $y = 0 \iff t^2 - 4t = 0 \iff t(t-4) = 0 \iff t = 0$ or 4. Also, $\big(x'(t), y'(t)\big) = \big(\frac{1}{2}(3t^2 - 6t), 2t - 4\big) = \big(\frac{3}{2}\,t(t-2), 2(t-2)\big)$ and so $x'(t) = 0 \iff t = 0$ or 2, and $y'(t) = 0 \iff t = 2$. There is a vertical point at $t = 0$, and there is a cusp when $t = 2$. We make a table of values and plot points.

t	x	y
-1	-2	5
0	0	0
1	-1	-3
2	-2	-4
3	0	-3
4	8	0



(b) The curve has a cusp (a sharp point). Find the slope of the curve at the cusp.

Solution: We have $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{2(t-2)}{\frac{3}{2}\,t(t-2)} = \dfrac{4}{3t}$, when $t \ne 2$. At the cusp, $t = 2$ and we have $\lim\limits_{t \to 2} \dfrac{dy}{dx} = \frac{2}{3}$.

(c) Find the area of the region which is bounded by the curve and the x-axis.

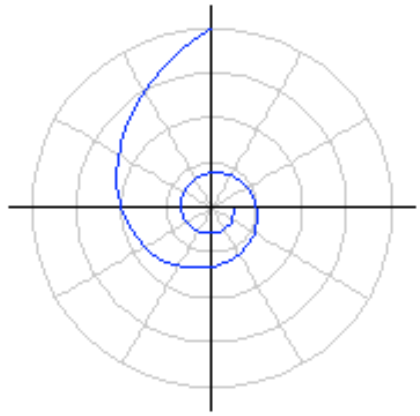
Solution: The area between the portion of the curve with $2 \le t \le 4$ and the x-axis (with $-2 \le x \le 8$) is $A_1 = -\displaystyle\int_{t=2}^{4} y(t)\,x'(t)\,dt$ (since $y(t)$ is negative), and the area between the portion of the curve with $0 \le t \le 2$ and the x-axis (with $-2 \le x \le 0$) is $A_2 = \displaystyle\int_{t=0}^{2} y(t)\,x'(t)\,dt$ (since $y(t)$ and $x'(t)$ are both negative). So the area of the desired region is $A = A_1 - A_2 = -\displaystyle\int_{t=0}^{4} y(t)\,x'(t)\,dt = -\int_0^4 (t^2 - 4t)\cdot\tfrac{1}{2}(3t^2 - 6t)\,dt =$ $-\displaystyle\int_0^4 \tfrac{3}{2}\,t^4 - 9t^3 + 12t^2\;dt = -\Big[\tfrac{3}{10}t^5 - \tfrac{9}{4}t^4 + 4t^3\Big]_0^4 = -\Big[t^3\big(\tfrac{3}{10}t^2 - \tfrac{9}{4}t + 4\big)\Big]_0^4 = -64\big(\tfrac{24}{5} - 9 + 4\big) = \tfrac{64}{5}$.

3: Consider the polar curve $r = \dfrac{2\pi}{\theta}$.

(a) Sketch the portion of the curve with $\frac{\pi}{2} \le \theta \le 4\pi$.

Solution: We make a table of values and sketch the curve.

θ	r
$\frac{\pi}{2}$	4
π	2
$\frac{3\pi}{2}$	$\frac{4}{3}$
2π	1
$\frac{5\pi}{2}$	$\frac{4}{5}$
3π	$\frac{2}{3}$
$\frac{7\pi}{2}$	$\frac{4}{7}$
4π	$\frac{1}{2}$



(c) Find the arclength of the portion of the curve with $\pi \le \theta \le 2\pi$.

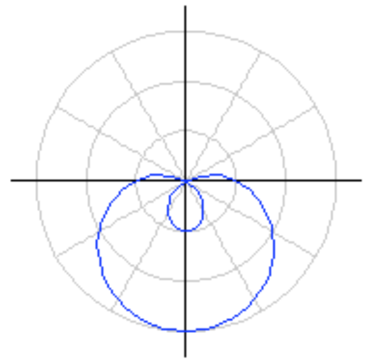
Solution: The arclength is $L = \displaystyle\int_{\theta=\pi}^{2\pi} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta = \int_{\pi}^{2\pi} \sqrt{\left(\frac{2\pi}{\theta}\right)^2 + \left(-\frac{2\pi}{\theta^2}\right)^2}\, d\theta = \int_{\pi}^{2\pi} 2\pi \sqrt{\frac{1}{\theta^2} + \frac{1}{\theta^4}}\, d\theta$ $= \displaystyle\int_{\pi}^{2\pi} \frac{2\pi\sqrt{\theta^2+1}}{\theta^2}\, d\theta$. Let $\tan\phi = \theta$ so that $\sec\phi = \sqrt{1+\theta^2}$ and $\sec^2\phi\, d\phi = d\theta$. Then $L = \displaystyle\int_{\theta=\pi}^{2\pi} \frac{2\pi \sec^3\phi\, d\phi}{\tan^2\phi}$ $= \displaystyle\int_{\theta=\pi}^{2\pi} \frac{2\pi\, d\phi}{\cos\phi \sin^2\phi} = \int_{\theta=\pi}^{2\pi} \frac{2\pi\, \cos\phi\, d\phi}{(1-\sin^2\phi)\sin^2\phi}$. Now let $u = \sin\phi$ so that $du = \cos\phi\, d\phi$. Then we have $L = 2\pi \displaystyle\int_{\theta=\pi}^{2\pi} \frac{du}{(1-u^2)\,u^2} = 2\pi \int_{\theta=\pi}^{2\pi} \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} + \frac{1}{u}\, du = 2\pi\left[\tfrac{1}{2}\ln|1+u| - \tfrac{1}{2}\ln|1-u| - \frac{1}{u^2}\right]_{\theta=\pi}^{2\pi} =$ $2\pi\left[\tfrac{1}{2}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{u^2}\right]_{\theta=\pi}^{2\pi} = 2\pi\left[\tfrac{1}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} = 2\pi\left[\tfrac{1}{2}\ln\left(\frac{(1+\sin\phi)^2}{\cos^2\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} =$ $2\pi\left[\ln\left(\frac{1+\sin\phi}{\cos\phi}\right) - \frac{1}{\sin\phi}\right]_{\theta=\pi}^{2\pi} = 2\pi\Big[\ln(\tan\phi + \sec\phi) - \csc\phi\Big]_{\theta=\pi}^{2\pi} = 2\pi\left[\ln\left(\theta + \sqrt{\theta^2+1}\right) - \frac{\sqrt{\theta^2+1}}{\theta}\right]_{\theta=\pi}^{2\pi}$ $= 2\pi\ln\left(2\pi + \sqrt{4\pi^2+1}\right) - \sqrt{4\pi^2+1} - 2\pi\ln\left(\pi + \sqrt{\pi^2+1}\right) + 2\sqrt{\pi^2+1}$.

4: Consider the polar curve $r = 1 - 2\sin\theta$.

(a) Sketch the curve.

Solution: We make a table of values and then sketch the curve. The curve is symmetric in the x-axis.

θ	r
$-\pi/2$	3
$-\pi/3$	$1+\sqrt{3}$
$-\pi/6$	2
0	1
$\pi/6$	0
$\pi/3$	$1-\sqrt{3}$
$\pi/2$	-1



(b) Find the area of the inner loop of the curve.

Solution: Note that since $r = 0$ when $\theta = \frac{\pi}{6}$, so that the line $\theta = \frac{\pi}{6}$ is tangent to one branch of the curve at the origin, the area of the small loop is given by $A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}(1-2\sin\theta)^2\,d\theta = \int_{\pi/6}^{\pi/2} 1-4\sin\theta+4\sin^2\theta\,d\theta = \int_{\pi/6}^{\pi/2} 1 - 4\sin\theta + 2 - 2\cos 2\theta\;d\theta = \Big[3\theta + 4\cos\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/2} = \frac{3\pi}{2} - \left(\frac{\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2}$.

5: A circular hoop of radius 1 is fixed in position centered at the origin $(0,0)$. Another hoop of radius 1, initially centered at $(2,0)$, rolls without slipping once around the first hoop.

(a) Find a parametric equation for the curve followed by the point on the second hoop which was initially at $(3,0)$.

Solution: The center of the second hoop follows the circle of radius 2 centered at $(0,0)$, so the position of the center of the second hoop is given by $(x,y)=(\cos t,\sin t)$. Note that as the second hoop rolls once around the first hoop, it must spin twice about its own center (so that it rolls without slipping), so the position of the point on the hoop which was initially at $(3,0)$ is given by $(x,y)=(2\cos t,2\sin t)+(\cos 2t,\sin 2t)=(2\cos t+\cos 2t,2\sin t+\sin 2t)$.

(b) Find the total distance travelled by the point on the second hoop.

Solution: The distance travelled is the same as the arclength of the curve, which is given by

$$\begin{aligned}
L &= 2\int_{t=0}^{\pi}\sqrt{x'(t)^2+y'(t)^2}\;dt \\
&= 2\int_0^{\pi}\sqrt{(-2\sin t-2\sin 2t)^2+(2\cos t+2\cos 2t)^2}\;dt \\
&= 4\int_0^{\pi}\sqrt{(\sin t+\sin 2t)^2+(\cos t+\cos 2t)^2}\;dt \\
&= 4\int_0^{\pi}\sqrt{\sin^2 t+2\sin t\sin 2t+\sin^2 2t+\cos^2 t+2\cos t\cos 2t+\cos^2 2t}\;dt \\
&= 4\int_0^{\pi}\sqrt{2+2\sin t\sin 2t+2\cos t\cos 2t}\;dt \\
&= 4\int_0^{\pi}\sqrt{2+2\sin t(2\sin t\cos t)+2\cos t(1-2\sin^2 t)}\;dt \\
&= 4\int_0^{\pi}\sqrt{2+4\sin^2 t\cos t+2\cos t-4\sin^2 t\cos t}\;dt \\
&= 4\int_0^{\pi}\sqrt{2+2\cos t}\;dt \\
&= 4\int_0^{\pi}\sqrt{4\cos^2(t/2)}\;dt \\
&= 8\int_0^{\pi}\cos(t/2)\;dt \\
&= \Big[16\sin(t/2)\Big]_0^{\pi} \\
&= 16
\end{aligned}$$