MATH 136 Solutions Assignment 1 Fall/05

September 24, 2005

Following are the solutions. Most of the solutions are given using MATLAB. However, you are not required to use MATLAB for these assignments.

Contents

1 page 11 #12 (use a check column)

5 marks

Solution. There is no solution. The last equation after elementary row operations has a nonzero on the right-hand-side but all zeros on the left. This can be seen in the file named output which is obtained from the following MATLAB program.

```
!rm output
diary output
echo on
A=[
1 -3 4 -4 -2
3 -7 7 -8 -5
-4 6 -1 7  8
]
A(2,:)=A(2,:)-3*A(1,:)
A(3,:)=A(3,:)+4*A(1,:)
A(2,:)=A(2,:)/2
A(3,:)=A(3,:)+6*A(2,:)
diary off
```

The file output contains the following:

```
echo on
A=[

A =


     1    -3     4    -4    -2
     3    -7     7    -8    -5
    -4     6    -1     7     8

1 -3 4 -4 -2
3 -7 7 -8 -5
-4 6 -1 7  8
]
A(2,:)=A(2,:)-3*A(1,:)
```

```
A =

     1    -3     4    -4    -2
     0     2    -5     4     1
    -4     6    -1     7     8

A(3,:)=A(3,:)+4*A(1,:)

A =

     1    -3     4    -4    -2
     0     2    -5     4     1
     0    -6    15    -9     0

A(2,:)=A(2,:)/2

A =

    1.0000   -3.0000    4.0000   -4.0000   -2.0000
         0    1.0000   -2.5000    2.0000    0.5000
         0   -6.0000   15.0000   -9.0000         0

A(3,:)=A(3,:)+6*A(2,:)

A =

    1.0000   -3.0000    4.0000   -4.0000   -2.0000
         0    1.0000   -2.5000    2.0000    0.5000
         0         0         0    3.0000    3.0000

diary off
```

■

2 page 11 #16 (use a check column)

5 marks

Solution. The system is consistent. The last line is all zeros, while the previous lines are in triangular form and provide a solution using e.g. backtracking; e.g. $x_4 = 0$ results in one solution after backtracking. This can be seen from the output of the following MATLAB file. (The check column is included.)

```
!rm output
diary output
echo on
A=[
1 0 0 -2 -3 -4
0 2 2 0   0  4
0 0 1 3  1   5
-2 3 2 1 5  9
]
A(4,:)=A(4,:)+2*A(1,:)
A(2,:)=A(2,:)/2
A(4,:)=A(4,:)-3*A(2,:)
A(4,:)=A(4,:)+A(3,:)
echo off
diary off
```

The output file contains the following:

```
A=[

A =

     1     0     0    -2    -3    -4
     0     2     2     0     0     4
     0     0     1     3     1     5
    -2     3     2     1     5     9

1 0 0 -2 -3 -4
0 2 2 0   0  4
0 0 1 3  1   5
```

```
-2 3 2 1 5  9
]
A(4,:)=A(4,:)+2*A(1,:)

A =

     1     0     0    -2    -3    -4
     0     2     2     0     0     4
     0     0     1     3     1     5
     0     3     2    -3    -1     1

A(2,:)=A(2,:)/2

A =

     1     0     0    -2    -3    -4
     0     1     1     0     0     2
     0     0     1     3     1     5
     0     3     2    -3    -1     1

A(4,:)=A(4,:)-3*A(2,:)

A =

     1     0     0    -2    -3    -4
     0     1     1     0     0     2
     0     0     1     3     1     5
     0     0    -1    -3    -1    -5

A(4,:)=A(4,:)+A(3,:)

A =

     1     0     0    -2    -3    -4
     0     1     1     0     0     2
     0     0     1     3     1     5
     0     0     0     0     0     0
```

```
echo off
```

■

---

3 page 12 #22

5 marks

Solution. The final row requires $5+3*h=0$, i.e. $h=-5/3$. This can be seen from the following MATLAB program and the output in the file output (a check column is included).

```
!rm output
diary output
echo on
h=sym('h')
A=[
2 -3 h (2-3+h)
-6 9 5 (-6+9+5)
]
A(1,:)=A(1,:)/2
A(2,:)=A(2,:)+6*A(1,:)
echo off
diary off
```

and the output file

```
h=sym('h')

h =

h


A=[
2 -3 h (2-3+h)
```

```
A =

[    2,   -3,    h, -1+h]
[   -6,    9,    5,    8]


-6 9 5 (-6+9+5)
]
A(1,:)=A(1,:)/2

A =

[          1,       -3/2,      1/2*h, -1/2+1/2*h]
[         -6,          9,          5,          8]


A(2,:)=A(2,:)+6*A(1,:)

A =

[          1,       -3/2,      1/2*h, -1/2+1/2*h]
[          0,          0,      5+3*h,      5+3*h]


echo off
```

■

---

4 page 12 #24

8 marks

Solution.

a. **TRUE.** From page 8, we get that two linear systems that are row equivalent have the same solution set.

b. **FALSE.** From page 7, two matrices are row equivalent if one can be transformed to the other using elementary row operations.

c. **FALSE.** From page 4, an inconsistent system is one which has no solutions.

d. **TRUE.** This is stated on page 3.
However, note that if we take a linear system and add a duplicate of one of the rows, then the solution set is unchanged. However, the augmented matrices of these two systems do not have the same number of rows and so cannot be *row* equivalent to one another.

■

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5 page 13 #33

5 marks

Solution.

$$\begin{aligned} T_1 &= (10 + 20 + T_2 + T_4)/4 \\ T_2 &= (T_1 + 20 + 40 + T_3)/4 \\ T_3 &= (T_4 + T_2 + 40 + 30)/4 \\ T_4 &= (10 + T_1 + T_3 + 30)/4 \end{aligned}$$

■

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6 page 13 #34

5 marks

Solution. The solution is given by the final line of the matlab program output.

```
!rm output
diary output
echo on
A= [
```

```
  4 -1 0 -1 30
 -1 4 -1 0 60
 0 -1 4 -1 70
-1 0 -1 4 40 ]
%%%%%%%% Forward reduction step (to row echelon form)
A=A([4 2 3 1],:)                % interchange rows 1 and 4
A(1,:)=-A(1,:)                  % multiply row 1 by -1
A(2,:)=A(2,:)+A(1,:)             % pivot using pivot element (1,1)
A(4,:)=A(4,:)-4*A(1,:)             % pivot using pivot element (1,1)
A(2,:)=A(2,:)/4                  % multiply row 2 by 1/4
A(3,:)=A(3,:)+A(2,:)             % pivot using pivot element (2,2)
A(4,:)=A(4,:)+A(2,:)             % pivot using pivot element (2,2)
A(4,:)=A(4,:)+A(3,:)             % pivot using pivot element (3,3)
%%%%%%%%%%%%% Backward substitution step to RREF
A(4,:)=A(4,:)/12                  % multiply row 2 by 1/4
A(3,:)=A(3,:)+2*A(4,:)             % pivot using pivot element (4,4)
A(2,:)=A(2,:)+A(4,:)             % pivot using pivot element (4,4)
A(1,:)=A(1,:)+4*A(4,:)             % pivot using pivot element (4,4)
A(3,:)=A(3,:)/4                  % multiply row 3 by 1/4
A(1,:)=A(1,:)-A(3,:)             % pivot using pivot element (3,3)
echo off
diary off
```

and the output file

```
A= [ 4 -1 0 -1 30; -1 4 -1 0 60; 0 -1 4 -1 70; -1 0 -1 4 40 ]

A =

     4    -1     0    -1    30
    -1     4    -1     0    60
     0    -1     4    -1    70
    -1     0    -1     4    40

%%%%%%%% Forward reduction step (to row echelon form)
A=A([4 2 3 1],:)                % interchange rows 1 and 4

A =
```

```
    -1     0    -1     4    40
    -1     4    -1     0    60
     0    -1     4    -1    70
     4    -1     0    -1    30

A(1,:)=-A(1,:)                      % multiply row 1 by -1

A =

     1     0     1    -4   -40
    -1     4    -1     0    60
     0    -1     4    -1    70
     4    -1     0    -1    30

A(2,:)=A(2,:)+A(1,:)                 % pivot using pivot element (1,1)

A =

     1     0     1    -4   -40
     0     4     0    -4    20
     0    -1     4    -1    70
     4    -1     0    -1    30

A(4,:)=A(4,:)-4*A(1,:)                 % pivot using pivot element (1,1)

A =

     1     0     1    -4   -40
     0     4     0    -4    20
     0    -1     4    -1    70
     0    -1    -4    15   190

A(2,:)=A(2,:)/4                      % multiply row 2 by 1/4

A =

     1     0     1    -4   -40
     0     1     0    -1     5
```

```
     0    -1     4    -1    70
     0    -1    -4    15   190

A(3,:)=A(3,:)+A(2,:)              % pivot using pivot element (2,2)

A =

     1     0     1    -4   -40
     0     1     0    -1     5
     0     0     4    -2    75
     0    -1    -4    15   190

A(4,:)=A(4,:)+A(2,:)              % pivot using pivot element (2,2)

A =

     1     0     1    -4   -40
     0     1     0    -1     5
     0     0     4    -2    75
     0     0    -4    14   195

A(4,:)=A(4,:)+A(3,:)              % pivot using pivot element (3,3)

A =

     1     0     1    -4   -40
     0     1     0    -1     5
     0     0     4    -2    75
     0     0     0    12   270

%%%%%%%%%%%%%% Backward substitution step to RREF
A(4,:)=A(4,:)/12                   % multiply row 2 by 1/4

A =

    1.0000         0    1.0000   -4.0000  -40.0000
         0    1.0000         0   -1.0000    5.0000
         0         0    4.0000   -2.0000   75.0000
```

```
         0         0         0    1.0000   22.5000

A(3,:)=A(3,:)+2*A(4,:)            % pivot using pivot element (4,4)

A =

    1.0000         0    1.0000   -4.0000  -40.0000
         0    1.0000         0   -1.0000    5.0000
         0         0    4.0000         0  120.0000
         0         0         0    1.0000   22.5000

A(2,:)=A(2,:)+A(4,:)              % pivot using pivot element (4,4)

A =

    1.0000         0    1.0000   -4.0000  -40.0000
         0    1.0000         0         0   27.5000
         0         0    4.0000         0  120.0000
         0         0         0    1.0000   22.5000

A(1,:)=A(1,:)+4*A(4,:)            % pivot using pivot element (4,4)

A =

    1.0000         0    1.0000         0   50.0000
         0    1.0000         0         0   27.5000
         0         0    4.0000         0  120.0000
         0         0         0    1.0000   22.5000

A(3,:)=A(3,:)/4                   % multiply row 3 by 1/4

A =

    1.0000         0    1.0000         0   50.0000
         0    1.0000         0         0   27.5000
         0         0    1.0000         0   30.0000
         0         0         0    1.0000   22.5000
```

```
A(1,:)=A(1,:)-A(3,:)              % pivot using pivot element (3,3)

A =

    1.0000         0         0         0   20.0000
         0    1.0000         0         0   27.5000
         0         0    1.0000         0   30.0000
         0         0         0    1.0000   22.5000

echo off
```

■

---

7 page 25 #2

4 marks

Solution. For the solution, use MATLAB, i.e. enter the matrix (call it A) and use rrefmovie(A). ■

8 page 25 #4

5 marks

Solution. For the solution, use MATLAB, i.e. enter the matrix (call it A) and use rrefmovie(A). ■

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9 Complex Linear System

Find the solution in C, the complex numbers, to the following linear system of equations

$$\begin{array}{rcrcl} (2+i)x_1 & - & (1+i)x_2 & = & -2+i \\ 2x_1 & + & 4ix_2 & = & 6i, \end{array}$$

where i denotes $\sqrt{-1}$. ***6 marks***

Solution. The solution is found using elementary row operations. We see this in the MATLAB program (a check column is included):

```
!rm output
diary output
echo on
k=sym('k')
A=[
2+i -(1+i) -2+i  (2+i-1-i-2+i)
2  4*i  6*i  (2+4*i+6*i)
]
A0=A(:,1:2);                        % save original augmented matrix
b0=A(:,3);                 % save original right-hand side
%%%%%%%%%%%%%%%%%%
A=A([2 1],:)               % exchange rows 1 and 2
A(1,:)=A(1,:)/2
A(2,:)=A(2,:)-(2+i)*A(1,:)
A(2,:)=A(2,:)/A(2,2)
A(1,:)=A(1,:)-A(1,2)*A(2,:)
disp('solution is')
x=A(:,3);
disp(num2str(x))
disp('verify solution')
A0*x-b0
echo off
diary off

A=[
2+i -(1+i) -2+i  (2+i-1-i-2+i)
2  4*i  6*i  (2+4*i+6*i)

A =

   2.0000 + 1.0000i  -1.0000 - 1.0000i  -2.0000 + 1.0000i  -1.0000 + 1.0000i
   2.0000                  0 + 4.0000i        0 + 6.0000i   2.0000 +10.0000i

]
```

```
A0=A(:,1:2);                       % save original augmented matrix
b0=A(:,3);                 % save original right-hand side
%%%%%%%%%%%%%%%%%%
A=A([2 1],:)               % exchange rows 1 and 2

A =

   2.0000                   0 + 4.0000i        0 + 6.0000i   2.0000 +10.0000i
   2.0000 + 1.0000i  -1.0000 - 1.0000i  -2.0000 + 1.0000i  -1.0000 + 1.0000i

A(1,:)=A(1,:)/2

A =

   1.0000                   0 + 2.0000i        0 + 3.0000i   1.0000 + 5.0000i
   2.0000 + 1.0000i  -1.0000 - 1.0000i  -2.0000 + 1.0000i  -1.0000 + 1.0000i

A(2,:)=A(2,:)-(2+i)*A(1,:)

A =

   1.0000                   0 + 2.0000i        0 + 3.0000i   1.0000 + 5.0000i
        0             1.0000 - 5.0000i   1.0000 - 5.0000i   2.0000 -10.0000i

A(2,:)=A(2,:)/A(2,2)

A =

   1.0000                   0 + 2.0000i        0 + 3.0000i   1.0000 + 5.0000i
        0             1.0000             1.0000             2.0000

A(1,:)=A(1,:)-A(1,2)*A(2,:)

A =

   1.0000                   0                  0 + 1.0000i   1.0000 + 1.0000i
        0             1.0000             1.0000             2.0000
```

```
disp('solution is')
solution is
x=A(:,3);
disp(num2str(x))
0+1i
1+0i
disp('verify solution')
verify solution
A0*x-b0

ans =

     0
     0

echo off
```

■

---

10 Bonus Question

10 marks

(you can use the *sym command* in MATLAB if you like)
Consider the linear system (in variables x, y, z, w)

$$\begin{array}{rcrcrcrcl} x & + & 2y & + & kz & + & w & = & 0 \\ 2x & + & 3y & - & 2z & + & (1-k)w & = & k \\ x & + & 2y & - & z & + & (2k+1)w & = & k+1 \\ kx & + & y & + & z & + & (k^2-2)w & = & 3 \end{array}$$

1. Find a value or values of k so that this system has no solution.

   Solution. We look at the final triangular form after the row eliminations below. Suppose that $k+1 \neq 0$. Then the last row after elementary row operations shows that there will be no solution if and only if

   $$9k^2 - 2k - 3 = 0 \quad \text{and} \quad k^2 + 3k + 2 = (k+2)(k+1) \neq 0.$$

This means that we use the roots of the quadratic equation and get:

$$\text{there is no solution if and only if: } k = \frac{1 \pm 2\sqrt{7}}{9}.$$

The MATLAB program used is:

```
!rm output
diary output
echo on
k=sym('k');
A=[
1 2 k 1
2 3 -2 1-k
1 2 -1 2*k+1
k 1 1 k^2-2];
b=[
0; k; k+1; 3];
%c= [A b]*ones(5,1);
A2=[A b ]
A3=A2;
%%% pivot column 1
A3(2,:)=A3(2,:)-2*A3(1,:)
A3(3,:)=A3(3,:)-A3(1,:)
A3(4,:)=A3(4,:)-k*A3(1,:)
%%% pivot column 2
A3(2,:)=-A3(2,:)
A3(4,:)=A3(4,:)-(1-2*k)*A3(2,:)
A3=simplify(A3)
A3(3,:)=-A3(3,:)
%%% pivot column 3
A3(4,:)=A3(4,:)-(A3(4,3)/A3(3,3))*A3(3,:)
%%%%%%%%%%%%%%%%%%%%%
A4=simplify(A3)
echo off
diary off
```

and the output file is:

```
k=sym('k');
A=[
1 2 k 1
2 3 -2 1-k
1 2 -1 2*k+1
k 1 1 k^2-2];
b=[
0; k; k+1; 3];
%c= [A b]*ones(5,1);
A2=[A b ]

A2 =

[     1,     2,     k,     1,     0]
[     2,     3,    -2,   1-k,     k]
[     1,     2,    -1, 2*k+1,   k+1]
[     k,     1,     1, k^2-2,     3]


A3=A2;
%%% pivot column 1
A3(2,:)=A3(2,:)-2*A3(1,:)

A3 =

[      1,      2,      k,      1,      0]
[      0,     -1, -2*k-2,   -1-k,      k]
[      1,      2,     -1,  2*k+1,    k+1]
[      k,      1,      1,  k^2-2,      3]


A3(3,:)=A3(3,:)-A3(1,:)

A3 =

[      1,      2,      k,      1,      0]
[      0,     -1, -2*k-2,   -1-k,      k]
[      0,      0,   -1-k,    2*k,    k+1]
```

```
[      k,      1,      1,  k^2-2,      3]


A3(4,:)=A3(4,:)-k*A3(1,:)

A3 =

[       1,       2,       k,       1,       0]
[       0,      -1,  -2*k-2,    -1-k,       k]
[       0,       0,    -1-k,     2*k,     k+1]
[       0,   1-2*k,   1-k^2, k^2-2-k,       3]


%%% pivot column 2
A3(2,:)=-A3(2,:)

A3 =

[       1,       2,       k,       1,       0]
[       0,       1,   2+2*k,     k+1,      -k]
[       0,       0,    -1-k,     2*k,     k+1]
[       0,   1-2*k,   1-k^2, k^2-2-k,       3]


A3(4,:)=A3(4,:)-(1-2*k)*A3(2,:)

A3 =

[                    1,                    2,                    k,
[                    0,                    1,                2+2*k,
[                    0,                    0,                 -1-k,
[                    0,                    0, 1-k^2-(1-2*k)*(2+2*k), k^


A3=simplify(A3)

A3 =
```

```
[            1,             2,             k,             1,             0]
[            0,             1,        2+2*k,           k+1,            -k]
[            0,             0,         -1-k,           2*k,           k+1]
[            0,             0, -1+3*k^2+2*k,       3*k^2-3,     3+k-2*k^2]


A3(3,:)=-A3(3,:)

A3 =

[            1,             2,             k,             1,             0]
[            0,             1,        2+2*k,           k+1,            -k]
[            0,             0,          k+1,          -2*k,          -1-k]
[            0,             0, -1+3*k^2+2*k,       3*k^2-3,     3+k-2*k^2]


%%% pivot column 3
A3(4,:)=A3(4,:)-(A3(4,3)/A3(3,3))*A3(3,:)

A3 =

[                                        1,
[                                        0,
[                                        0,
[                                        0,


%%%%%%%%%%%%%%%%%%%%%
A4=simplify(A3)

A4 =

[           1,            2,            k,            1,            0]
[           0,            1,       2+2*k,          k+1,           -k]
[           0,            0,         k+1,         -2*k,         -1-k]
[           0,            0,           0, 9*k^2-2*k-3,   k^2+3*k+2]
```

```
echo off
```

■

2. Find a value or values of k so that this system has infinitely many solutions.

   Solution. If $k+1 \neq 0$, then we see that we either get no solutions (as above in part 1) or a unique solution when the quadratic $9k^2 - 2k - 3 \neq 0$.

   Therefore, we now consider the case when $k + 1 = 0$. This yields a unique solution $w = 0$. However, z can be chosen arbitrarily, i.e. we have an infinite number of solutions. ■

3. For each value of k found in Item 2, find all solutions (in vector form).

   Solution. We can use backsubstitution to obtain the general solution $w = 0$, z arbitrary, $y = -k = 1$, $x = 0 - 2y - kz = -2 + z$, i.e. the general solution is of the form

$$\begin{pmatrix} -2+z \\ 1 \\ z \\ 0 \end{pmatrix}.$$

■