ASSIGNMENT 1 ANSWERS

(Q1) i)

Table 1: $1/n$ as n increases

n	$\frac{1}{n}$
1	1
2	0.5
4	0.25
10	0.1
100	0.01
1000	0.001
10000	0.0001

ii) As n increrases $1/n$ decreases toward zero.
iii) Since n is a natural number and the numerator has a constant value of 1 any negative value for L would work. However if we were interested in the 'best' lower bound, it seems natural to choose $L = 0$ since we are getting closer and closer to zero as n increases, but we never get exactly there.

(Q2) i)

Table 2: $1/n^2$ as n increases

n	$\frac{1}{n}$
1	1
2	0.25
4	0.0625
10	0.01
100	0.0001
1000	0.000001
10000	0.00000001

ii) As n increrases $1/n^2$ decreases toward zero. The decrease is **faster** than for $1/n$.
iii) The same argument as in Q1 gives $L = 0$.

(Q3) i) By choosing $n = 10$, 100, 1000 and higher powers of 10 we get that

$$\frac{(-1)^n}{n}$$

gets closer and closer to zero. This is because by choosing **even** n the negative 1 in the numerator goes away. What about when n is odd? Well, the distance from zero is given by the absolute value

and this does not change when the numerator is negative.
ii) The best we can do *for all* n is to pick $L_1 = -1$ and $L_2 = 1/2$, which can be seen by looking at the first two terms ($n = 1$ and $n = 2$) and noting that the sequence decreases in distance from 0.
iii) Thus the difference between this case and the two previous ones is that we can't choose lower bounds that reflect what happens for large n.

(Q4)

Table 3: $1/x$ as x gets smaller, $x > 0$

$\frac{1}{x}$	x
1	1
2	0.25
4	0.0625
10	0.01
100	0.0001
1000	0.000001
10000	0.00000001

Table 4: $1/x$ as x gets smaller, $x < 0$

$\frac{1}{x}$	x
-1	1
-2	-0.25
-4	-0.0625
-10	-0.01
-100	-0.0001
-1000	-0.000001
-10000	-0.00000001

The above tables tell us that not only does the distance between $1/x$ and zero grow very large as x gets smaller and smaller, but the sign of $1/x$ changes depending on the sign of the input.

(Q5) The point of the exercise is to notice that when m is a fraction with π in the numerator then $\sin(mn)$ will eventually repeat itself (see table 5 for detailed numbers). When π does not appear in the numerator then $\sin(mn)$ will not repeat itself. This is because π is irrational.

(Q6) i) $x^3 + x^2 = x^2(x + 1) = 0$ so $x = 0$ and $x = -1$ are the two solutions. $x = 0$ is a so-called double root.
ii) $x^2 + 6x + 7 = 0$ can be solved by the quadratic formula to yield

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 7}}{2}$$

Table 5: $\sin mn$ for various m as n changes

n	$m = 1$	$m = 2\pi$	$m = \frac{\pi}{6}$
1	0.8415	0	0.5
2	00.9093	0	0.8660
3	0.1411	0	1.0
4	-0.7568	0	0.8660
5	-0.9589	0	0.5
6	-0.2794	0	0
7	0.6570	0	-0.5
8	0.9894	0	-0.8660
9	0.4121	0	-1
10	-5440	0	-0.8660
11	-1	0	-0.5
12	-0.5366	0	0
13	0.4202	0	0.5

which simplifies to

$$x = -3 \pm \sqrt{2}.$$

iii) Using the hint factor $x^4 + 6x^3 + 6x^2 - 6x - 7 = (x^2 - 1)(x^2 + 6x + 7)$. Thus two answers will be

$$x = -3 \pm \sqrt{2}.$$

from part ii). For the other two we must have $x^2 - 1 = 0$ or $x = \pm 1$.

(Q7) i) The distance between x and 4 is less than 0.1. Or $3.9 < x < 4.1$
ii) The distance between x^2 and 9 is less than 0.01. Or $8.99 < x^2 < 9.01$. To find the bounds on x we need to remember that squaring a negative number gives positive numbers. So we have either

$$\sqrt{8.99} < x < \sqrt{9.01}$$

or

$$-\sqrt{9.01} < x < -\sqrt{8.99}$$

iii) The distance between x^3 and 8 is less than 0.001. This time however things are easier because the third power of a negative number is still negative. Hence $7.999 < x^3 < 8.001$ means (this time I give approximate numerical values):

$$1.99991666319420 < x < 2.00008332986135$$

As a final comment notice that even though I picked the numbers inside the absolute values to look like things should simplify ($9 = 3^2$, $8 = 2^3$) the combination of the numbers inside the absolute value and the bound on distance meant that things did not work out 'nice' (they rarely do in practice).

(Q8) This simple looking equation is hard to solve. If you have a graphing calculator you can eyeball where the two graphs meet. Otherwise you could start at $x = 0$ and increase x by a set amount, say 0.01. For each guess compute $\sin(x)$ and $x/2$ and see how close the guess is to providing an answer. Once you knew roughly where the answer was you could take smaller steps (0.001, then 0.0001 and so on). This is not something you'd want to do by hand, though programming it might be easy.

(Q9) In one step the input is x and the output is $\sin(x^2)$. In two steps the input is again x and the first output is x^2. The second input is then $z = x^2$ and the second output is $\sin(z)$. There are many choices to show that the two steps cannot be reversed in order. Consider $x = \pi$, hence $x^2 = \pi^2$ and $\sin(x^2) = -0.43030121700009$. If we reverse the order then we find that $\sin\pi = 0$ and $0^2 = 0$, a very different answer.

(Q10) The equation $x^2 + y^2 = 2^2$ defines a circle of radius 2 centered at the origin. The equation does not define a function because for any x there are two possible values of y and hence we do not have a single output for a single input.

(Q11) The graph of the exponential can be found in your text. You should find that the output at -5 is tiny while at 5 is huge. The quantity I ask you to compute looks a lot like the slope of a line (rise over run) and you should find that for the exponential this "rate of change" is the value of the exponential itself.

$$\frac{e^{2x}}{e^{-3x}}e^{x^2} = \exp(x^2 - x)$$