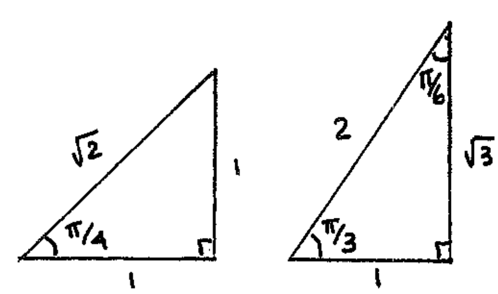
UNIVERSITY OF WATERLOO
FINAL EXAMINATION
WINTER TERM 2004

COURSE NUMBER	MATH 128
COURSE TITLE	Calculus 2
DATE OF EXAM	April 14, 2004
TIME PERIOD	7:00 - 10:00 p.m.
DURATION OF EXAM	3 hours
NUMBER OF EXAM PAGES (including this cover sheet , & one for ROUGH WORK)	12 pages
INSTRUCTORS (please circle your instructor)	L001 E. M. Moskal L002 J. Munroe L003 B.J. Marshman L006 J. West
EXAM TYPE	Closed book
ADDITIONAL MATERIALS ALLOWED	NONE

The following identities and triangles may be useful:

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$



Student Name ______________________

Student ID Number ______________________

Notes:

1. Fill in your name, ID number and sign the paper.
2. Answer all questions in the space provided. Continue on the back of the preceding page if necessary. Show **ALL** your work.
3. Any questions totalling 100 marks constitute a complete examination.
4. Check that the examination has 12 pages.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions. Reference any theorems used by their names or acronyms.**

Marking Scheme:

Question	Mark	Out of
1		15
2		12
3		10
4		10
5		10
6		10
7		13
8		10
9		10
10		10
Total		100 + 10

[15] 1. a) Evaluate the definite integral $\int_0^{\pi/2} x \cos x \, dx$.

b) Find the general antiderivative $\int \frac{3x+4}{x^2-4} \, dx$.

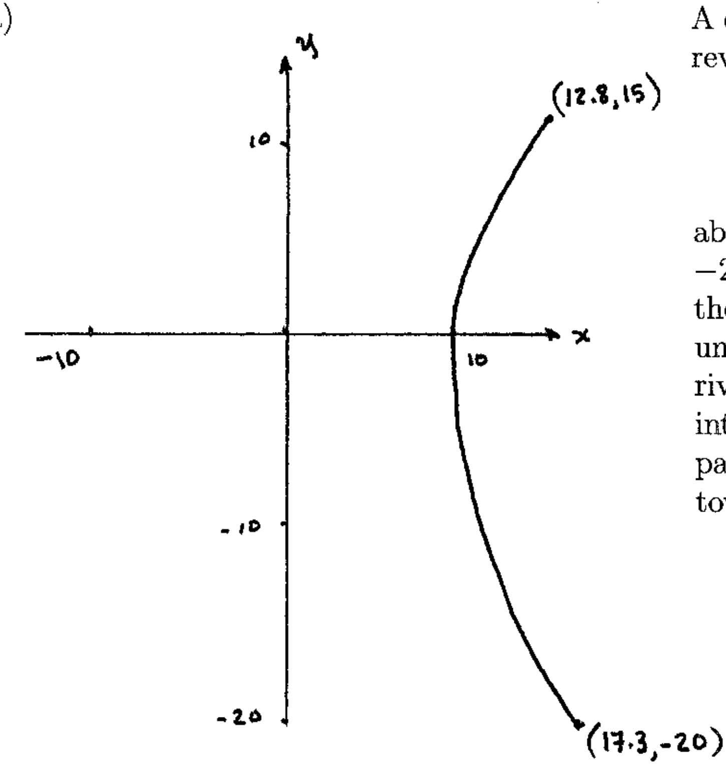
c) Determine whether the improper integral $\int_0^{\infty} xe^{-x} \, dx$ converges or diverges, and explain what this tells you about the series $\sum_{n=0}^{\infty} \frac{n}{e^n}$.

d) Find the indicated partial derivatives for $f(x,y) = e^{2x} \sin(x^2 + y)$

(i) $f_x(0, \pi/3)$

(ii) $f_{xy}(0, \pi/3)$

[12] 2. a)

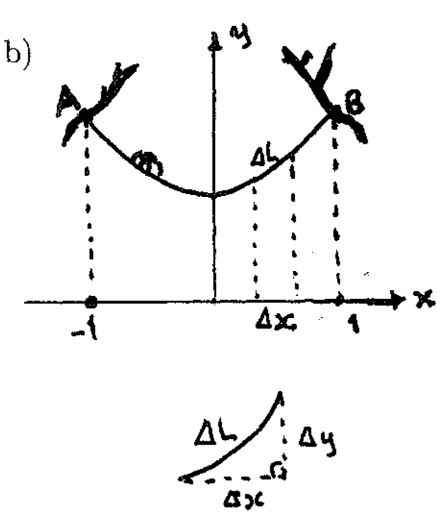


A cooling tower is formed by revolving the hyperbola

$$x = \sqrt{100 + \frac{y^2}{2}}$$

about the y-axis for $-20 \leq y \leq 15$. Sketch the tower and a suitable volume element ΔV, then derive and evaluate a definite integral representing the capacity (total volume) of the tower.

b)



A strand of spider web hangs between two branches, from A to B, in the shape of the curve $y = \frac{1}{2}(e^x + e^{-x})$. Derive a definite integral, with limits, which represents how far the spider crawls along the web from A to B. (Start with the element of arc ΔL as shown. DO NOT evaluate the integral.)

[10] 3. a) Solve the linear initial value problem $y' + \frac{2}{x}y = \frac{\pi}{x^2}\cos(\pi x),\ y(1) = 1.$

b) In a chemical reaction, a substrate S with concentration y is consumed according to the separable DE

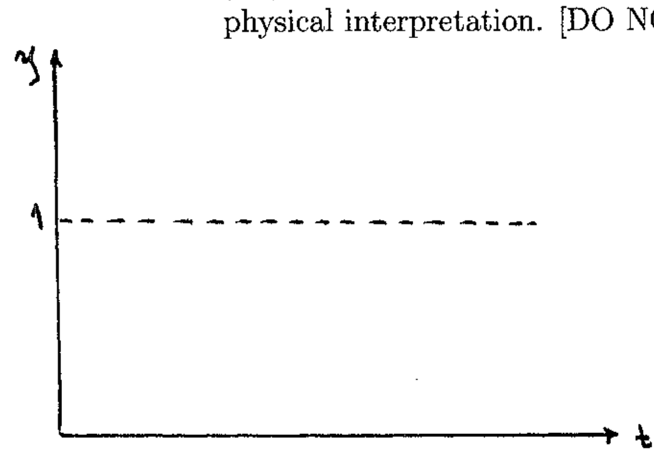
$$\frac{dy}{dt} = -\frac{y}{1+y} \quad , \quad \text{where } y \geq 0.$$

(i) Find the general solution of this separable DE. (Leave your answer in implicit form.)

(ii) If substrate is added at rate $R = 0.05$, the DE becomes

$$\frac{dy}{dt} = -\frac{y}{1+y} + 0.05.$$

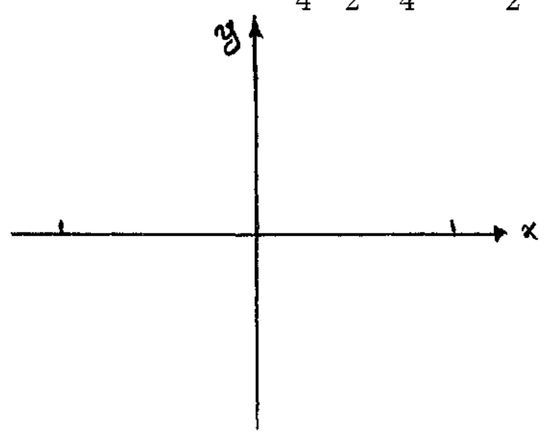
Show that $\frac{dy}{dt} > 0$ for $0 \leq y < 1$, and $\frac{dy}{dt} < 0$ for $y > 1$, and sketch qualitative graphs of the solution for each case. State the equilibrium solution and give a physical interpretation. [DO NOT attempt to solve the DE.]



[10] 4. a) In the movie A Beautiful Mind, genius John Nash rides his bike in a figure 8 as he thinks. Suppose his path is given by

$$\mathbf{x}(t) = (\cos t,\, \sin 2t), \quad 0 \le t \le 2\pi.$$

(i) Sketch his path in the x-y plane. Indicate the direction, and label the points where $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$. [DO NOT ATTEMPT to find an x-y equation.]



(ii) Find his velocity $\mathbf{v}(t)$, and sketch the vectors $\mathbf{v}(0)$ and $\mathbf{v}(\frac{\pi}{2})$ on your graph in (i).

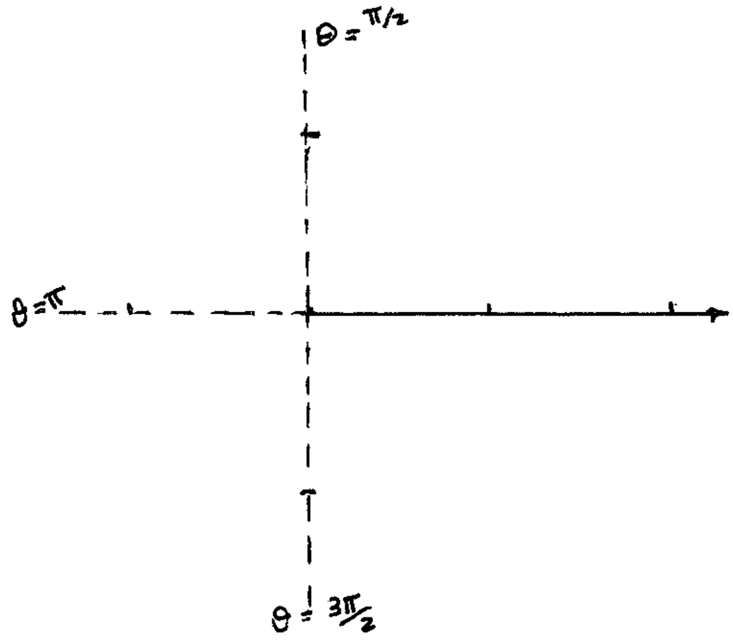
(iii) Note that $\mathbf{v}(\frac{\pi}{4}) = \mathbf{v}(\frac{3\pi}{4})$. Are there any other pairs t_1, t_2 in $[0, 2\pi]$ where $\mathbf{v}(t_1) = \mathbf{v}(t_2)$? Explain, using your graph.

b) A marble rolls down a spiral trough, following the path

$$\mathbf{x}(t) = \left(\cos(t^2), \sin(t^2), 2\pi - \frac{1}{2}t^2\right), \; 0 \le t \le \sqrt{4\pi}.$$

Find the marble's speed $||\mathbf{v}(t)||$, and use this to find the distance travelled, ie., the length of the path.

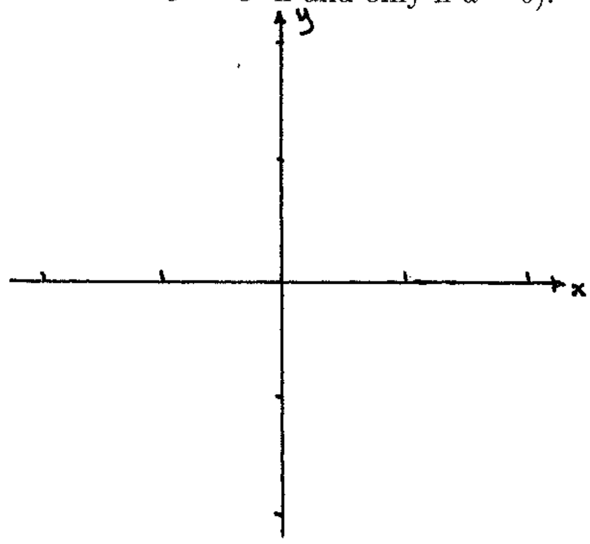
[10] 5. a) Show that the spiral $r = \theta$ intersects $r = \pi + \sin\theta$ at $\theta = \pi$, and make a careful sketch of these polar graphs, labelling the points where $\theta = 0,\ \pi/2,\ \pi,\ 3\pi/2$, and 2π on both curves.



b) Derive and evaluate the integral needed to find the area between the two curves for $0 \le \theta \le \pi$, starting with a suitable area element ΔA which you should show on your sketch.

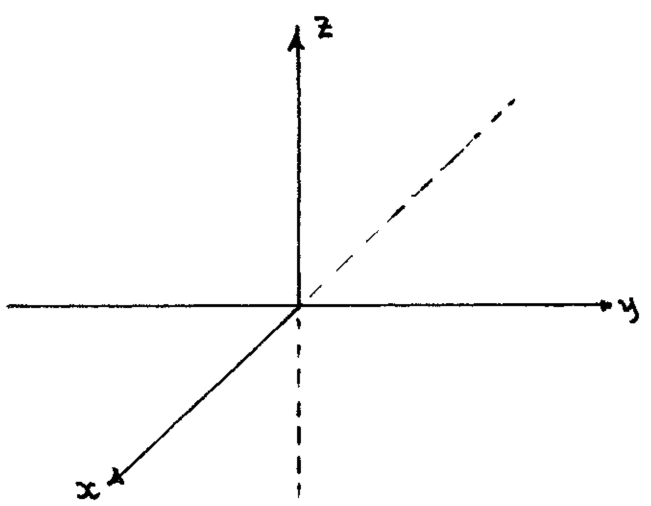
[10] 6. Consider the function $f(x,y) = e^{2-(x^2+y^2)}$.

a) Sketch the level sets $f(x,y) = c$ for $c = e, 1$, and e^{-2} on the given axes. (Recall that $e^a = e^b$ if and only if $a = b$).



b) Explain why the range of f is $(0, e^2]$ for all (x,y) in $\mathbb{R}^2$.

c) Sketch the surface $z = f(x,y)$, showing the cross-sections $x = 0$, $y = 0$ and $z = 1$.



d) Find the equation of the tangent plane to $z = f(x,y)$ at the point $(1,1,1)$.

e) The relative change in altitude is $\frac{\Delta z}{z}$. What is the approximate value of $\frac{\Delta z}{z}$ between $(x,y) = (1,1)$ and $(x,y) = (1.1, 0.8)$? [Use the increment form for Δz.]

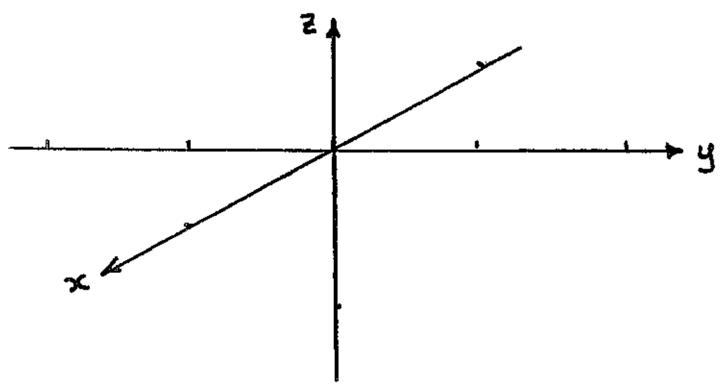
[13] 7. This problem investigates a pond and its water temperature.

a) The basin (bottom) of the pond has the shape given by

$$z = f(x,y) = -100 + \left(\frac{x}{10}\right)^2 + \left(\frac{y}{20}\right)^2,$$

where distances are measured in metres.

i) Given that the top surface (water level) is at $z = 0$, sketch the basin of the pond, showing the cross-sections $x = 0$, $y = 0$, and $z = 0$.



ii) A catfish crawls around the basin of the pond at a constant depth of $z = -20$m. Find the equation of its path.

iii) In what direction is the slope of the basin steepest at the point $(10, 40, -95)$?

b) Suppose the water temperature in the pond is given by

$$T(x,y,z) = 20 + 0.02x - 0.01y + 0.1\,z \quad {}^\circ\text{C}.$$

(i) If the catfish heads vertically upward in the z-direction, with x and y fixed, will it experience an increase or decrease in temperature?

(ii) A swimmer on the surface of the pond follows the path $\mathbf{x}(t) = (t^2,\ t^3, 0)$. What rate $\frac{dT}{dt}$ of change in the water temperature T does she experience at $t = 1$?

(iii) What is the directional derivative of T in the tangential direction $\mathbf{x}'(1)$ to the swimmer's path at the point in (ii)?

[10] 8. Consider the following three infinite series

$$\text{A: } \sum_{n=0}^{\infty} \frac{n}{3n+1} \qquad \text{B: } \sum_{n=0}^{\infty} \pi \frac{(-2)^n}{5^{n-1}} \qquad \text{C: } \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$$

a) Only one of these series diverges. Use the N^{th} Term Test to determine which one.

b) Determine whether the remaining two series converge absolutely or conditionally, using the comparison test and the alternating series test.

c) Find the sum of the series that converges absolutely.

[10] 9. a) Find the interval of convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

b) Use a known series to find a Maclaurin series for $\frac{1}{1+t^2}$, and then use a suitable theorem to show that the Maclaurin series for $\arctan x$ is precisely the series in a).

c) Note that $x = 1$ in the series gives $\arctan 1$, or $\pi/4$. How many terms would you need to estimate π with $|\text{error}| < 0.001$?

[10] 10. a) State the Maclaurin series for $f(x) = \sin x$, and use it to write down the Taylor polynomial $P_4(x)$ for $\sin x$ about $a = 0$.

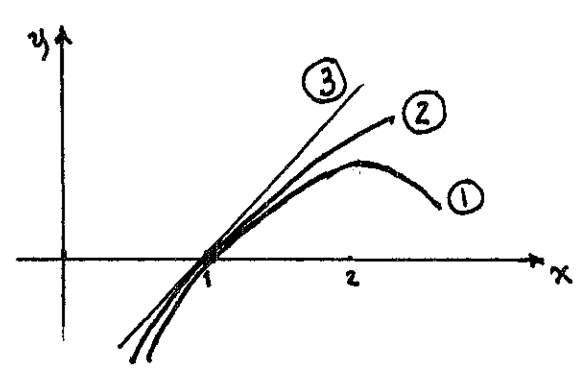
b) On an assignment, you showed that

$$|\sin x - \left(x - \frac{x^3}{3!}\right)| \leq \frac{|x|^5}{5!} \quad \text{for} \quad x \in \mathbb{R}.$$

Use $P_4(x)$ and the given inequality to find an approximate value for $\sin(0.1)$, and an upper bound on the error.

c) Re-state the inequality given in b) in 'big-O' notation, and use it to evaluate $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$.

d) On the sketch, one of the curves is $y = \ln x$, one is its Taylor polynomial $P_1(x)$ centered at 1, and one is $P_2(x)$. Identify which is which, giving reasons, and explain which of $P_1(1.5)$ or $P_2(1.5)$ you would expect to give the better approximation of $\ln(1.5)$, and why.



ROUGH WORK