1. i) Given the path of a particle $(x(t), y(t))$ state the formula for $l(T)$, the arclength travelled by the particle on the time interval $0 \le t \le T$.

2

$$\ell(T) = \int_{t=0}^{t=T} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t=0}^{t=T} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

ii) Show that for all T at which the particle has non-zero velocity, the arclength increases as a function of T.

by FTC II $\quad \frac{d\ell}{dT} = \frac{d}{dT}\left(F(T) - F(0)\right)$

where $\frac{dF(\square)}{d\square} = \sqrt{x'(\square)^2 + y'(\square)^2}$

3

$\therefore \frac{d\ell}{dt} = \sqrt{x'(T)^2 + y'(T)^2}$ so $\frac{d\ell}{dt} > 0$

unless $\sqrt{x'(T)^2 + y'(T)^2} = 0$ this can only happen if $x'(T) = y'(T) = 0$ or $\vec{v}(T) = \vec{0}$

iii) Sketch the curve $(x(t), y(t)) = (2\sin(t), \cos(t))$ for $0 \le t \le 2\pi$. Find the position at $t = \pi/6$ and the tangent line at $t = 5\pi/6$ and add them to your sketch.

Notice $\frac{x(t)^2}{(\sqrt{2})^2} + y(t)^2 = \sin^2(t) + \cos^2(t) = 1$ for all t

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$\therefore$ the curve is an ellipse.

The tangent vector is given by $\frac{d\vec{x}}{dt} = (2\cos t, -\sin t)$

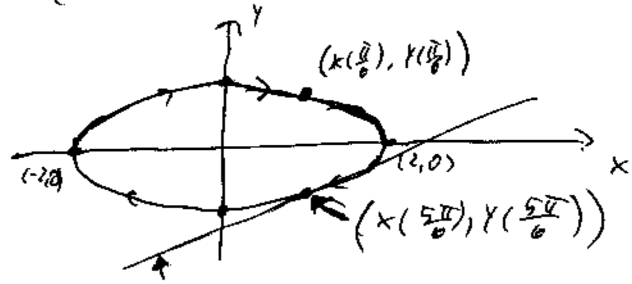
at $t = \frac{\pi}{6}$ $\quad x(t) = 2\sin\left(\frac{\pi}{6}\right) = 2 \cdot \frac{1}{2} = 1$

$y(t) = \cos(t) = \frac{\sqrt{3}}{2} \approx 0.85$

at $t = \frac{5\pi}{6}$ $\quad x(t) = 2\sin\left(\frac{5\pi}{6}\right) = 1$; $y(t) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} \approx -0.85$

$x'(t) = 2\cos(t) \approx -1.7$; $y'(t) = -\sin(t) = -1/2$

tangent line $\quad (x(s), y(s)) = (1, -0.85) + s(-1.7, -0.5)$



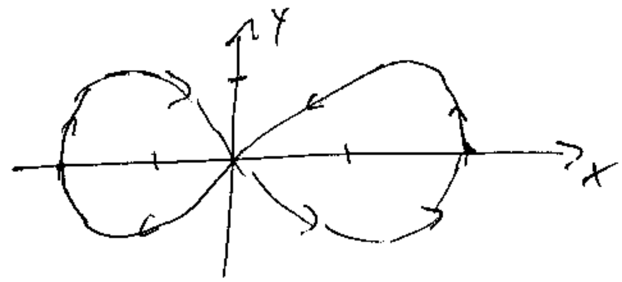
1. iv) Sketch $(x(t), y(t)) = (2\cos(t), \sin(2t))$ on $0 \le t \le 2\pi$ and state where and when the curve crosses itself.

Notice that at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ $\sin(2t) = 0$
and at $t = \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos(t) = 0$

at $t=0$ start at $(x(t), y(t)) = (2, 0)$

Now $x(t)$ has a period of 2π
$y(t)$ has a period of π

4



v) Sketch the curve given by $(x(t), y(t)) = ((t-4)^2, (t-4)^2)$ on $0 \le t \le 8$ and explain why a tangent line cannot be found at $t = 4$.

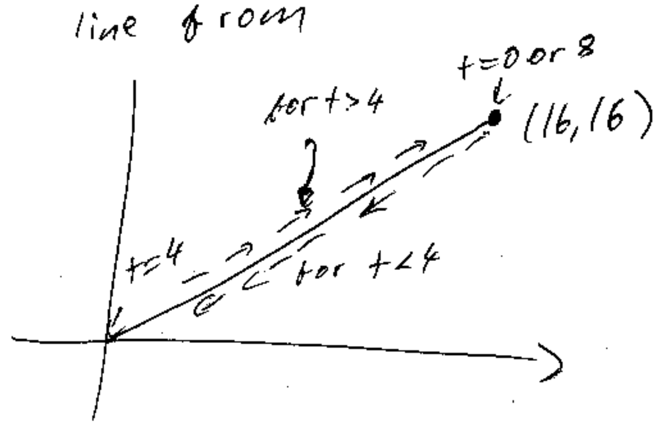
Notice that $y(t) = x(t)$

$x(0) = x(8) = 16$

$x'(t) = 4(t-4)^3 = y'(t)$ so at
$t = 4$ $(x'(t), y'(t)) = (0, 0)$

4

Since speed $= 0$ there is no velocity vector to compute the tangent line from



2. i) Use a partial fractions decomposition to show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ has partial sums $S_N = 1 - \frac{1}{N+1}$.

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ if $A+B=,$ $A=1$

3

$S_1 = 1 - \frac{1}{2}$

$= \frac{1}{n} - \frac{1}{n+1}$

$S_2 = \frac{1}{2} - \frac{1}{3} + S_1 = 1 - \frac{1}{3}$

$S_N = \frac{1}{N} - \frac{1}{N+1} + S_{N-1}$ & by recursi

$S_3 = \frac{1}{3} - \frac{1}{4} + S_2 = 1 - \frac{1}{4}$

$= 1 - \frac{1}{N+1}$

ii) Define what it means to say $\sum_{n=1}^{\infty} a_n$ converges, and hence determine whether or not the series in i) converges.

down 1 inch

If $S_N = \sum_{n=1}^{N} a_n$ then

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{N\to\infty} S_N$ exists

3

For the above

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1 - 0 = 1$

and the series converges

iii) Does

$$\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^2+4}}$$

converge?

down 1 inch

Here $a_n = \frac{n+3}{\sqrt{n^2+4}} = \frac{1+3/n}{\frac{1}{n}\sqrt{n^2+4}} = \frac{1+\frac{3}{n}}{\sqrt{\frac{n^2}{n^2}+\frac{4}{n^2}}}$

$= \frac{1+\frac{3}{n}}{\sqrt{1+\frac{4}{n^2}}}$

3

Thus as $n\to\infty$ $a_n \to \frac{1+0}{\sqrt{1+0}} = 1$

Since $a_n \not\to 0$ as $n\to\infty$ the Nth term test gives us that

$\sum_{n=1}^{\infty} \frac{n+3}{\sqrt{n^2+4}}$ diverges

$n^2+4 = 5, 9, 15, \ldots$
$5n^2 = 5, 20, 45, \ldots$

2. iv) Does

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

converge?

(2) hard one

Here $a_n = \frac{n}{n^2+4} \geq \frac{1}{5}\frac{n}{n^2} = \frac{1}{5n}$ for all n

but by the p-test $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\therefore$ by comparison test $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges

could also do integral test!

v) Does

$$\sum_{n=1}^{\infty} e^{-2n^2}$$

converge?

(4)

Here $a_n = e^{-2n^2} \leq e^{-n}$ for all n

Next let $f(x) = e^{-x}$ so that $f(n) = e^{-n}$

$f'(x) = -e^{-x} < 0$ for $x \geq 1$ so $f(\cdot)$ is decreasing

Also $\int_1^{\infty} f(x)dx = \lim_{L\to\infty} \int_1^L e^{-x}dx = \lim_{L\to\infty} -e^{-x}\Big|_1^L = \frac{1}{e}$ $\therefore$ $\sum_{n=1}^{\infty} e^{-n}$ converges by the integral test and $\sum_{n=1}^{\infty} e^{-2n^2}$ converges by the comparison test

vi) Show that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

converges and find N so that the error in approximating the true sum by s_N is less than 0.0001

(5)

Notice that the above is an alternating series with $P_n = \frac{1}{\sqrt{n}}$ so that $P_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = P_n$ for all n. Furthermore $P_n \to 0$ as $n \to \infty$ and thus by the Alternating Series Test $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

Next recall that writing $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} = S_N + R_N$ we have $|R_N| < P_{N+1}$. We want $|R_n| < 0.0001 = 10$

$P_{N+1} = \frac{1}{\sqrt{N+1}} < 10^{-4}$, squaring $\frac{1}{N+1} < 10^{-8}$

$N+1 > 10^8$ $\therefore$ $N > 10^8 - 1$ will do the trick

3. i) State the precise definition of the statement that the sequence a_n tends to L.

$a_n \to L$ as $n \to \infty$ provided that for any $\varepsilon > 0$ we can find N so that $n > N$ guarantees

$$|a_n - L| < \varepsilon$$

2

ii) Use the above definition to prove that

$$a_n = \frac{1}{\exp(n)}$$

tends to 0.

Consider one $\varepsilon > 0$, we want to find N so that $n > N$ means

5

$$|a_n - 0| < \varepsilon \quad \text{or} \quad e^{-n} < \varepsilon$$

because $e^{-n} > 0$ for all $n > 0$

take $\ln$ of both sides to get

$$-n < \ln \varepsilon \quad \text{or} \quad n > -\ln \varepsilon$$

Thus for any $\varepsilon > 0$ choose N to be the first positive integer bigger than $-\ln \varepsilon$

Since we can do this for any $\varepsilon > 0$ we get $a_n \to 0$

iii) Show rigorously that if $a_n \longrightarrow 1$ and $b_n \longrightarrow 3$ then $c_n = a_n + b_n \longrightarrow 4$ (i.e. prove the Limit Sum Rule). HINT: you will need to use the triangle inequality which states that for any two number A and B, $|A+B| \le |A| + |B|$.

For some $\varepsilon > 0$

$a_n \to 1$ means we can find N^a so that $n > N^a$ guarantees $|a_n - 1| <$

$b_n \to 3$ means we can find N^b so that $n > N^b$ guarantees $|b_n - 3| <$

3

Now let $N = \max\{N^a, N^b\}$ so that for $n > N$ we have

$|c_n - 4| = |a_n + b_n - 4| = |a_n - 1 + b_n - 3|$

$\le |a_n - 1| + |b_n - 3|$ by triangl inequali

that $|a_n - 1| < \frac{\varepsilon}{2}$, $|b_n - 3| < \frac{\varepsilon}{2}$ so $|c_n - 4| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $n > N$

iv) Consider the sequence 0.9, 0.99, 0.999, 0.9999, ... Define a_n for this sequence and show (no need for a rigorous proof) that the limit of the sequence is 1.

down 2inches

We can write ~~then~~

$a_1 = 1 - \frac{1}{10}$

$a_2 = 1 - \frac{1}{100} = 1 - \frac{1}{10^2}$

$a_3 = 1 - \frac{1}{1000} = 1 - \frac{1}{10^3}$

3

$\vdots$

$a_n = 1 - \left(\frac{1}{10}\right)^n$

Now we know for $|\alpha| < 1$ $\alpha^n \to 0$ as $n \to$

so $a_n \to 1 - 0 = 1$ as $n \to \infty$

4. Give examples for each of the following:

i) A **series** that converges but does not converge absolutely.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

ii) A **sequence** a_n for which $|a_n|$ converges but a_n does not.

Let $a_n = (-1)^n$ then $|a_n| = 1$

and clearly $|a_n| \to 1$ as $n \to \infty$

but $a_n \not\to$ any number as $n \to \infty$

iii) A **series** with terms that increase when $1 \le n \le 10$ but which converges.

Let $a_n = \begin{cases} n & 1 \le n \le 10 \\ \frac{1}{n^2} & n > 10 \end{cases}$

iv) Two **sequences** a_n and b_n which both diverge but for which $c_n = a_n + b_n$ converges.

Let $a_n = n$ $\quad b_n = -n$

then $c_n = n - n = 0$ for all n

v) A **series**

$$\sum_{n=1}^{\infty} a_n$$

for which the sequence of partial sums S_N is decreasing but for which the sum diverg

This is a trick, really. Because if

$S_N > 0$ and decreasing then S_N has a lim

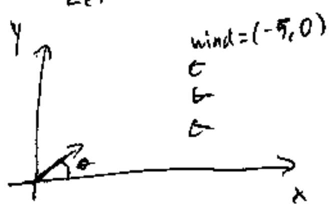
Clearly S_N must not have a lower bou

so $\sum_{n=1}^{\infty} -n$ will work

5. Consider a particle fired to the west from the origin at an angle of θ from the horizontal with a speed of 10 m s^{-1}, $g = 10 \text{ m s}^{-2}$.

i) If there is a horizontal wind with speed 5 m s^{-1} from the west find the [illegible] range of the particle. Assume you can neglect air resistance. Formulate your problem clearly in terms of physical laws and differential equations and include a sketch.

Let the x-axis point westward



$\vec{F}_g = (0, -mg)$ is the only f[orce]

neglect friction, Earth's rotati[on] etc. to get Newton's Law in vector form

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F} \quad \text{or} \quad \begin{cases} m\dfrac{d^2x}{dt^2} = 0 \\ m\dfrac{d^2y}{dt^2} = -mg \end{cases}$$

Now $x(0) = y(0) = 0$

and we are given

$\vec{V}(0) = (10\cos\theta, 10\sin\theta) + (-5, 0)$

$x'(0) = 10\cos\theta$

$y'(0) = 10\sin\theta$

Integrate

$x(t) = 10\cos\theta\, t - 5t$

$y(t) = 10\sin\theta\, t - \frac{g}{2}t^2$

ii) For what value of θ does the particle return to its launch site when it lands?

down 3 inches

We need the horizontal position at $t_{landing}$ to equal zero

or $20\sin\theta\cos\theta - 10\sin\theta = 0$

$= 10(2\cos\theta - 1)\sin\theta = 0$

$\sin\theta = 0$ or $\theta = 0, \pi$

or $2\cos\theta - 1 = 0$

or $\cos\theta = \frac{1}{2}$ $\boxed{\theta = \frac{\pi}{3}}$

so given $g = 10$

$y = 0$ when $t = 0$

or $10\sin\theta - \frac{10}{2}t = 0$

$t_{landing} = 2\sin\theta$

$x(t_{landing}) = 10 \cdot 2\sin\theta\cos\theta - 5 \cdot 2\sin\theta$

$= 10\sin(2\theta) - 10\sin\theta$

~~$\frac{d\,x(t_{landing})}{d\theta} = 20\cos(2\theta) - 10\sin\theta$~~

$= 0$

~~when $20\cos(2\theta) - 10\sin\theta$~~

5. iii) State the integral for the work done by the force of gravity on the particle during its tim in flight. Use any symmetry you find in the problem to simplify the integral.

Recall $$\text{work} = \int_{t=0}^{t=T} \vec{F} \cdot \frac{d\vec{s}}{dt}\, dt$$

Here $\vec{F} = \vec{F}_g = (0, -mg)$ so

$$\text{Work} = \int_{t=0}^{t=t_{landing}} -mg \frac{dy}{dt}\, dt$$

$$= -mg \int_{t=0}^{t=t_{landing}} \frac{dy}{dt}\, dt = -mg\, y(t)\Big|_{t=0}^{t=t_{la}}$$

$$= -mg\,(0-0) = 0$$

Alternatively

$\frac{dy}{dt} > 0$ $\quad 0 < t < \frac{t_{landing}}{2}$

$\frac{dy}{dt} < 0$ $\quad \frac{t_{landing}}{2} < t < t_{landing}$

$\therefore$ work = 0

iv) Now assume that you cannot neglect air resistance (or friction in other words). Use energ arguments to briefly explain whether the maximum range of the particle is increased (decreased.

In the above problem Energy is conserved. If friction is not neglecte Energy is lost to heat during flight. No matter how we model the friction the total energy decreases & hence the range of the particle must decrease as well.