White Test Paper

2 marks each

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1. For the following short answer questions (multiple choice, fill in the blank, and true/false), you do not need to justify your answer.

   (a) Which of the following trigonometric substitutions transforms

   $$\int \frac{x^3}{\sqrt{16-x^2}}\,dx \text{ to } -64\int \cos^3\theta\,d\theta\,?$$

   Circle the correct answer:

   i. $x = \cos\frac{\theta}{4}$
   ii. $x = \frac{1}{4}\cos\theta$
   iii. $x = \cos 4\theta$
   iv. $x = 4\cos\theta$

   (b) What is the complete set of k values such that $y = \cos kx$ satisfies the DE $y'' + y = 0$?

   Circle the correct answer:

   i. $k = \pm 1$
   ii. $k = 0$
   iii. $k = 0, \pm 1, \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$
   iv. $k = 0, \pm 1, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

   (c) What are the equilibrium solutions to $y' = x(y^2 + y - 2)$? $y^* = -2, 1$

   Circle the correct answer:

   (d) $\int_{-3}^{0} \frac{-dx}{x+1} = -\ln 2$ TRUE FALSE

   (e) $\int_0^\infty \sin x\,dx$ is divergent. TRUE FALSE

   (f) $\frac{dT}{dt} = \frac{-T+70}{10}$ is correctly classified as both separable and linear. TRUE FALSE

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2. Consider the parametric curve defined by $(x, y) = (2\sin t, 2\cos t)$, for $\pi \leq t \leq 2\pi$.

   (a) Calculate the arc length of the curve for $\pi \leq t \leq 2\pi$.

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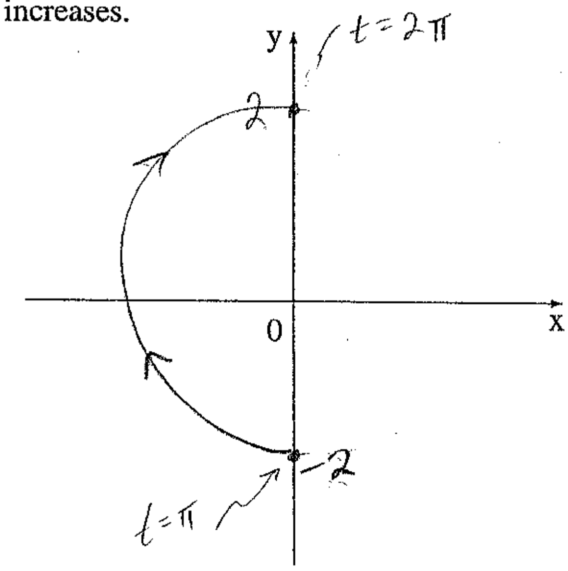
$$\frac{dx}{dt} = 2\cos t \quad (1)$$
$$\frac{dy}{dt} = -2\sin t \quad (1)$$

formula (1)

$$L = \int_{\pi}^{2\pi} \sqrt{(2\cos t)^2 + (-2\sin t)^2}\, dt$$
$$= \int_{\pi}^{2\pi} \sqrt{4\cos^2 t + 4\sin^2 t}\, dt$$
$$= \int_{\pi}^{2\pi} \sqrt{4(\cos^2 t + \sin^2 t)}\, dt \quad (1)$$
$$(1) \quad = 2t\Big]_{\pi}^{2\pi} = 4\pi - 2\pi = \boxed{2\pi} \quad (1)$$

   (b) Sketch the curve in the xy–plane. Label the curve where $t = \pi$ and $t = 2\pi$ along with the associated x and y values. Indicate with an arrow on the curve the direction in which the curve is traced as t increases.

t	x	y
π	0	-2
$\frac{5\pi}{4}$	$-\sqrt{2}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	2	0
$\frac{7\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$
2π	0	2



(1) - curve
(1) - arrows
(1) - labels

   (c) Determine the Cartesian equation of the curve.

$$x^2 + y^2 = 4\sin^2 t + 4\cos^2 t = 4(\sin^2 t + \cos^2 t) = 4$$

(2) or $\boxed{x^2 + y^2 = 4}$

   (d) For which t values in the given interval, $[\pi, 2\pi]$, does the curve have a vertical tangent(s)?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin t}{2\cos t} = \frac{-\sin t}{\cos t}, \quad \frac{dy}{dx} \to \infty \Rightarrow \boxed{\frac{dx}{dt} \to 0}: \quad (1)$$

$$\Rightarrow \cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots, \quad \text{so, } \boxed{t = \frac{3\pi}{2}} \quad (1)$$

   (e) For which t values in the given interval, $[\pi, 2\pi]$, does the curve have a horizontal tangent(s)?

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{-\sin t}{\cos t}, \quad \frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dt} = 0}: \quad (1)$$

$$\Rightarrow \sin t = 0 \Rightarrow t = 0, \pm\pi, \pm 2\pi, \ldots, \quad \text{so, } \boxed{t = \pi \;\&\; 2\pi} \quad (1)$$

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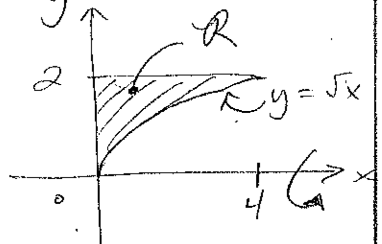
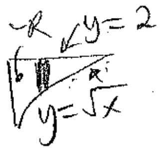
3. Let $\mathcal{R}$ be the region bounded between $y = \sqrt{x}$, the line $y = 2$, and the y–axis.

(a) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the x–axis.

$\sqrt{x} = 2 \Rightarrow x = 4$
(intersection)

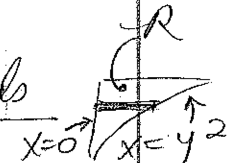


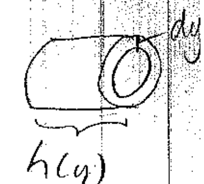


washers

OR

cylindrical shells





$$\left.\begin{aligned} r_o(x) &= 2 \\ r_i(x) &= \sqrt{x} \\ dV &= \pi(r_o^2 - r_i^2)\,dx \\ V &= \pi\int_0^4 (4 - x)\,dx \end{aligned}\right\}$$

4 — formula for V, integral

$= \pi\left[4x - \frac{x^2}{2}\right]_0^4$ 1

$= \pi\left(16 - \frac{16}{2}\right) = \boxed{8\pi}$ 1

$r(y) = y$

$h(y) = y^2 - 0 = y^2$

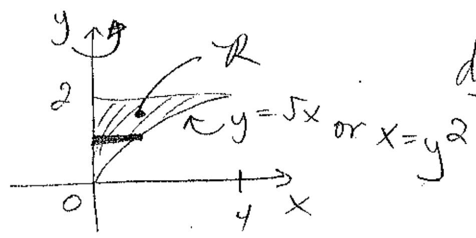
$$\left.\begin{aligned} dV &= 2\pi y h\,dy \\ V &= 2\pi\int_0^2 y(y^2)\,dy \\ &= 2\pi\int_0^2 y^3\,dy \end{aligned}\right\}$$

4 — formula for V, integral

$= 2\pi\left[\frac{y^4}{4}\right]_0^2$ 1

$= 2\pi\left(\frac{16}{4}\right) = \boxed{8\pi}$ 1

(b) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the y–axis.



disks:

$$\left.\begin{aligned} dV &= \pi r^2\,dy \\ V &= \pi\int_0^2 (y^2)^2\,dy \\ &= \pi\int_0^2 y^4\,dy \end{aligned}\right\}$$

2 — formula for V, integral

$= \pi\left[\frac{y^5}{5}\right]_0^2$ 1

$\therefore \boxed{V = \frac{32}{5}\pi}$ 1 white test paper.

4. Solve the initial value problem (IVP) $\frac{dP}{dt} = \sqrt{tP}$, $P(0) = 9$, where $P \geq 0$ and $t \geq 0$.

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$\frac{dP}{dt} = \sqrt{tP} = \sqrt{t}\sqrt{P}$, separable:

$\frac{dP}{\sqrt{P}} = \sqrt{t}\, dt$

(1) $\int P^{-\frac{1}{2}}\, dP = \int t^{\frac{1}{2}}\, dt$

(1) $2P^{\frac{1}{2}} = \frac{2}{3}t^{\frac{3}{2}} + c$, $\underline{\underline{c \geq 0}}$ (✩) (since $t \geq 0$ and $P \geq 0$)

$P^{\frac{1}{2}} = \frac{1}{3}t^{3/2} + \frac{c}{2}$ (← could apply IC here)

(1) $P = \left(\frac{1}{3}t^{3/2} + \frac{c}{2}\right)^2$

Apply IC: $P(0) = 9 = \left(0 + \frac{c}{2}\right)^2 = \frac{c^2}{4}$

$\Rightarrow c^2 = 36$

$\Rightarrow c = \pm 6 \longrightarrow c = 6$ (by ✩) (1)

$\therefore \boxed{P = \left(\frac{1}{3}t^{3/2} + 3\right)^2}$ (1)

5. Solve the IVP $y' = -x + y$, $y(0) = 2$.

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$y' = -x + y$ → not separable, check linear:

$\Rightarrow y' - y = -x$, linear: $\boxed{P(x) = -1}$ (1)

$\Rightarrow \boxed{I(x) = e^{\int P dx} = e^{\int -1\, dx} = e^{-x}}$ (1)

$\Rightarrow e^{-x}y' - e^{-x}y = -xe^{-x}$

$\Rightarrow \frac{d}{dx}\left(e^{-x}y\right) = -xe^{-x}$ (1)

$\Rightarrow \int \left[\frac{d}{dx}\left(e^{-x}y\right)\right] dx = \int -xe^{-x}\, dx$

IBP: let $u = -x$ $\quad v = -e^{-x}$

$du = -dx$ $\quad dv = e^{-x}dx$

LHS $\left(\frac{1}{2}\right)$ RHS $\left(1\frac{1}{2}\right)$

$\Rightarrow \boxed{e^{-x}y} = xe^{-x} - \int e^{-x}dx = \boxed{xe^{-x} + e^{-x} + C}$

$\Rightarrow \boxed{y = x + 1 + Ce^{x}}$ (1)

Apply IC: $y(0) = 2 = 0 + 1 + Ce^{0} = 1 + C$ (1)

$\Rightarrow C = 1$

$\therefore \boxed{y = x + 1 + e^{x}}$ (1)