Math 138 Physics Based Section Assignment 2

(Q1) Find the Laplace transform, $F(s)$ of each of the following functions (a is a parameter):
i) $f(t) = t^2 + 2t$
From the notes we know that $\mathcal{L}(t) = 1/s^2$ so that $\mathcal{L}(2t) = 2/s^2$ and that

$$\frac{d}{dt}t^2 = 2t$$

so that using the derivative rule we have

$$\mathcal{L}(2t) = \mathcal{L}(\frac{dt^2}{dt}) = s\mathcal{L}(t^2) - 0$$

where the last zero is due to the fact that $t^2 = 0$ at $t = 0$. Now use what we know about the transform of $2t$ to get

$$\mathcal{L}(t^2) = \frac{2}{s^3}.$$

Thus

$$\mathcal{L}(f(t)) = \frac{2}{s^3} + \frac{2}{s^2}$$

ii) $f(t) = t\exp{(-at)}$ using the rule for multiplying by the exponential from the notes
We know that if $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(\exp{(-at)}f(t)) = F(s+a)$. Now we also know that $\mathcal{L}(t) = 1/s^2$ so that

$$\mathcal{L}(f(t)) = \frac{1}{(s+a)^2}$$

iii) $f(t) = t\exp{(-at)}$ using the transform of $\exp{(-at)}$ and differentiating with respect to a
Now we know that

$$\mathcal{L}(\exp{(-at)}) = \frac{1}{s+a} = (s+a)^{-1}$$

and differentiating with respect to a gives

$$\mathcal{L}(-t\exp{(-at)}) = -(s+a)^{-2}$$

so that

$$\mathcal{L}(f(t)) = \frac{1}{(s+a)^2}$$

iv) $f(t) = t\sin{(at)}$
Here we know that

$$\mathcal{L}(\cos{(at)}) = \frac{s}{s^2+a^2} = s(s^2+a^2)^{-1}$$

and that using Chain Rule

$$\frac{d}{da}\cos(at) = -t\sin(at)$$

along with

$$\frac{d}{da}[s(s^2+a^2)^{-1}] = -2sa(s^2+a^2)^{-2}$$

so that finally we get

$$\mathcal{L}(f(t)) = \frac{2sa}{(s^2+a^2)^2}$$

v) $f''(t)$ for a general function $f(t)$ (call the transform of $f(t)$ $F(s)$).
We know that if $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}(\frac{d}{dt}f(t)) = sF(s) - f(0)$$

so that

$$\mathcal{L}(\frac{d^2}{dt^2}f(t)) = s\mathcal{L}(\frac{d}{dt}f'(t)) - f'(0)$$

but we already know the transform of $f'(t)$ so that

$$\mathcal{L}(\frac{d^2}{dt^2}f(t)) = s(sF(s) - f(0) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

vi) $H(t-1)$ where $H(.)$ is the Heaviside step function.
Here we are dealing with a discontinuous function that outputs the value 0 when $t < 1$ and the value 1 when $t > 1$. This suggest that we should return to the definition of the Laplace transform

$$\mathcal{L}(H(t-1)) = \int_0^\infty H(t-1)\exp(-ts)dt = \int_1^\infty \exp(-ts)dt$$

but the last integral can be written as the transform of the constant function 1, minus the part of the integral between $t = 0$ and $t = 1$, or

$$\int_1^\infty \exp(-ts)dt = \mathcal{L}(1) - \int_0^1 \exp(-ts)dt.$$

But $\mathcal{L}(1) = 1/s$ and

$$-\int_0^1 \exp(-ts)dt = \frac{-1+\exp(-s)}{s}$$

so that

$$\mathcal{L}(H(t-1)) = \frac{1}{s} - \frac{1}{s} + \frac{\exp(-s)}{s} = \frac{\exp(-s)}{s}$$

vii) $tH(t-1)$ where $H(.)$ is the Heaviside step function.
Here we again use the same trick as above, namely

$$\mathcal{L}(tH(t-1)) = \mathcal{L}(t) - \int_0^1 t\exp(-ts)dt.$$

You can do the last integral by parts to get

$$\int t\exp(-ts)dt = \frac{t\exp(-ts)}{-s} + \frac{1}{s}\int \exp(-ts)dt$$

or

$$\int t\exp(-ts)dt = -\frac{t\exp(-ts)}{s} - \frac{\exp(-ts)}{s^2}$$

which means

$$-\int_0^1 t\exp(-ts)dt = -\frac{1}{s^2} + \exp(-s)\frac{s+1}{s^2}.$$

This means that

$$\mathcal{L}(tH(t-1)) = \frac{1}{s^2} - \frac{1}{s^2} + \exp(-s)\frac{s+1}{s^2} = \exp(-s)\frac{s+1}{s^2}.$$

viii) $t(1-H(t-1))$ where $H(.)$ is the Heaviside step function.
We use the last part here along with the linearity of the Laplace transform to get:

$$\mathcal{L}(t(1-H(t-1)) = \mathcal{L}(t) - \mathcal{L}(tH(t-1)) = \frac{1}{s^2} - \exp(-s)\frac{s+1}{s^2}$$

ix) $\cos^2(3t)$
Here we use a trigonometric formula, namely

$$\cos(2M) = \cos^2(M) - \sin^2(M) = 2\cos^2(M) - 1$$

which holds for all M, even $M = 3t$. So that

$$\mathcal{L}(\cos^2(3t)) = \mathcal{L}((1/2) + \cos(6t)/2) = \frac{1}{2s} + \frac{s}{s^2+6^2}.$$

(Q2) Find the inverse Laplace transform using the method of partial fractions or by converting to a known form (use Q1 and your notes).
i)

$$F(s) = \frac{3s+2}{s^2+5s+6}$$

First factor the denominator as $s^2+5s+6 = (s+2)(s+3)$. Then use Partial Fractions to get

$$\frac{3s+2}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s+3A+2B}{s^2+5s+6}$$

which gives the two equations $(A+B) = 3$ and $3A+2B = 2$ which gives $A = -4$ and $B = 7$. So that

$$\frac{3s+2}{s^2+5s+6} = \frac{-4}{s+2} + \frac{7}{s+3}$$

and since

$$\mathcal{L}(\exp(-at)) = \frac{1}{s+a}$$

we have that the inverse Laplace transform is

$$-4\exp(-2t) + 7\exp(3t)$$

ii)

$$F(s) = \frac{3s+2}{s^3+5s^2+6s}$$

Here the denominator factors in a similar way as $s^3+5s^2+6s = s(s+2)(s+3)$. We can apply Partial Fractions again but this time we get three terms and three variables:

$$\frac{3s+2}{s^3+5s^2+6s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}.$$

When we recombine the three terms and equate we get that

$$0s^2+3s+2 = A(s+2)(s+3)+Bs(s+3)+Cs(s+2) = (A+B+C)s^2+(5A+3B+2C)s+6A$$

This can be solved to give $A = 1/3$, $B = 2$ and $C = -7/3$ and the inverse Laplace transform thus is

$$\frac{1}{3} + 2\exp(-2t) - \frac{7}{3}\exp(3t)$$

iii)

$$F(s) = \frac{\exp(-s)}{s}$$

You've done this one above in Q1 v). So the inverse Laplace transform is $H(t-1)$
iv)

$$F(s) = \frac{\exp(-s)}{(s+3)}$$

This is a tough one. You might be able to guess the answer though. First recall that

$$\mathcal{L}(f(t)\exp(-3t)) = F(s+3)$$

and that $\mathcal{L}(1) = 1/s$ so that

$$\mathcal{L}(\exp(-3t)) = \frac{1}{s+3}$$

Based on this and the previous part you might try to compute

$$\mathcal{L}(\exp(-3t)H(t-1)) = \mathcal{L}(\exp(-3t)) - \int_0^1 \exp(-(3+s)t)dt = \frac{1}{s+3} - \frac{1}{s+3} + \frac{\exp(-s-3)}{s+3}$$

or

$$\mathcal{L}(\exp(-3t)H(t-1)) = \exp(-3)\frac{\exp(-s)}{s+3}$$

which is only a factor of $\exp(3)$ off. Thus

$$\mathcal{L}(\exp(3)\exp(-3t)H(t-1)) = \frac{\exp(-s)}{s+3}$$

as required.
v)

$$F(s) = \frac{s+4}{(s+4)^2+3^2}$$

This one just tests whether you remember the transform of cosine and the rule for multiplying by an exponential. Since

$$\mathcal{L}(\cos(3t)) = \frac{s}{s^2+3^2}$$

we have

$$\mathcal{L}(\exp(-4t)\cos(3t)) = \frac{s+4}{(s+4)^2+3^2}.$$

Transforms like this are important because they often turn up in the study of damped oscillations (like a mass on a spring with friction). vi)

$$F(s) = \frac{3}{s^2+8s+25}$$

This one is just like the last one, BUT I didn't factor it out all nice for you. After a bit of trial and error, or more systematically by completing the square, I find:

$$s^2+8s+25 = s^2+8s+16+9 = (s+4)^2+3^2$$

so that upon recalling that

$$\mathcal{L}(\sin(3t)) = \frac{3}{s^2+3^2}$$

we find

$$\mathcal{L}(\exp(-4t)\sin(3t)) = \frac{3}{(s+4)^2+3^2}.$$

as desired.

(Q3) Solve the following initial value problems for $y(t)$ using the Laplace transform. The general strategy is as outlined in class:

1. Take Laplace transform of the DE and use the initial conditions
2. Simplify for $Y(s)$
3. If needed use partial fractions on the expression
4. Using tables of Laplace transforms (your notes, for example) find $y(t)$

i) $y' + y = 4$, $y(0) = 3$ Take the Laplace transform of the differential equation and let $Y(s) = \mathcal{L}(y(t))$:

$$sY(s) - y(0) + Y(s) = \frac{4}{s}$$

or

$$(s+1)Y(s) = \frac{4}{s} + 3$$

thus

$$Y(s) = \frac{4}{s(s+1)} + \frac{3}{s+1}.$$

The second term is easily recognized as an exponential, but the first must be broken up with the method of partial fractions:

$$\frac{4}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{(A+B)s + A}{s(s+1)}$$

which gives two equations $A = 4$ and $A + B = 0$ or $B = -4$. This means

$$Y(s) = \frac{4}{s} - \frac{4}{s+1} + \frac{3}{s+1} = \frac{4}{s} - \frac{1}{s+1}$$

and the inverse Laplace transform now gives

$$y(t) = 4 - \exp(-t).$$

Note that $y'(t) = \exp(-t)$ so that $y' + y = 4$ as desired.
ii) $y' + y = 4$, $y(0) = -3$ This one works almost exactly the same way so it let's us look at what role the initial conditions play in the problem. After Laplace transforming we get:

$$(s+1)Y(s) = \frac{4}{s} - 3$$

or

$$Y(s) = \frac{4}{s(s+1)} - \frac{3}{s+1}.$$

The first term is the same as in part i) and so the same partial fraction splitting up will work:

$$Y(s) = \frac{4}{s} - \frac{4}{s+1} - \frac{3}{s+1} = \frac{4}{s} - \frac{7}{s+1}$$

and this gives us the solution:

$$y(t) = 4 - 7\exp(-t).$$

iii) $y' + y = t$, $y(0) = 3$ Again taking the Laplace transform gives

$$(s+1)Y(s) = \frac{1}{s^2} + 3$$

or

$$Y(s) = \frac{1}{s^2(s+1)} + \frac{3}{s+1}.$$

We must break up the first term using partial fractions:

$$\frac{1}{s^2(s+1)} = \frac{As+B}{s^2} + \frac{C}{s+1} = \frac{(As+B)(s+1)+Cs^2}{s^2(s+1)}$$

The numerator of the last expression is $(A+C)s^2+(A+B)s+B$ which must equal $0s^2+0s+1$. This gives three equations $B=1$, $A+B=0$ which gives $A=-1$ and $A+C=0$ which gives $C=1$. This let's us rewrite $Y(s)$ as

$$Y(s) = \frac{-s+1}{s^2} + \frac{1}{s+1} + \frac{3}{s+1} = \frac{1}{s^2} - \frac{1}{s} + \frac{4}{s+1}$$

which can be easily inverted to give

$$y(t) = t - 1 + 4\exp(-t)$$

iv) $y' + y = \exp(-t)$, $y(0) = 3$ Again taking the Laplace transform

$$(s+1)Y(s) = \frac{1}{s+1} + 3$$

and so

$$Y(s) = \frac{1}{(s+1)^2} + \frac{3}{s+1}$$

Now from page 13 of the notes we have if $F(s) = \mathcal{L}(f(t))$ then $F'(s) = \mathcal{L}(-tf(t))$. Here

$$\mathcal{L}(-\exp(-t)) = \frac{-1}{s+1} = -(s+1)^{-1}$$

taking the derivative with respect to s gives $\mathcal{L}(t\exp(-t)) = (s+1)^{-2}$. Now we can invert $Y(s)$ to get

$$y(t) = t\exp(-t) + 3\exp(-t).$$

Notice how the exponential appeared by itself and with a t multiple out front.
v) $y' + y = \exp(-2t)$, $y(0) = 3$ Taking the Laplace transform gives

$$(s+1)Y(s) = \frac{1}{s+2} + 3$$

or

$$Y(s) = \frac{1}{(s+2)(s+1)} + \frac{3}{s+1}.$$

The result is easy to find using partial fractions

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{3}{s+1}$$

Taking the common denominator of the first two terms gives

$$0s + 1 = (A+B)s + A + 2B$$

or $B = 1$ and $A = -1$, thus

$$Y(s) = \frac{-1}{s+2} + \frac{1}{s+1} + \frac{3}{s+1}$$

and

$$y(t) = -\exp(-2t) + 4\exp(-t).$$

(Q4) Consider the mass, M, of a spherical planet of radius R, problem. The density as a function of radius is given by $d(r)$.
i) Find the mass of a planet with $d(r) = (1 + 0.05\sin(\frac{r}{R}))$.
Here

$$M(R) = \int_0^R 4\pi r^2 d(r) dr$$

Using the definition of $d(r)$ get

$$M(R) = \frac{4}{3}\pi R^3 + 4\pi 0.05 \int_0^R r^2 \sin(\frac{r}{R}) dr$$

Now using Maple I get

$$M(R) = \frac{4}{3}\pi R^3 + 4\pi 0.05 R^3(\cos(1) + 2\sin(1) - 2)$$

ii) Find the mass of a planet for which $d(r) = 1 - 0.25H(r - 0.5R)$.
Here we have to split the integral into two:

$$M(R) = \int_0^{R/2} 4\pi r^2 dr + \int_{R/2}^R 4\pi r^2 0.75 dr$$

and after all the algebra clears is

$$M(R) = \frac{25}{24}\pi R^3$$

iii) For a general $d(r)$ the mass can be written as a function of the planet's radius as $M(R)$. Find $M'(R)$ and explain what it means.
Here

$$M(R) = \int_0^R 4\pi r^2 d(r) dr$$

and using the results from the last assignment we find

$$M'(R) = 4\pi R^2 d(R)$$

so the rate of change of mass increases as R^2. This is because the larger the radius, the larger the volume of a spherical shell of thickness ΔR.

(Q5) Consider a mass m attached to a linear spring with spring constant k. The mass is placed on a frictionless table. i) Use Newton's law to derive the simple harmonic oscillator equation, or SHO, that governs the motion of the mass. Let $x(t)$ be the position of the mass and let the origin of your coordinates be placed at the position the mass occupies when the spring is not stretched.

Here the only force is $F = -kx(t)$ and Newton's second law, or in other words the equation of motion, reads

$$m\frac{d^2x}{dt^2} = -kx$$

define the frequency as

$$\omega = \sqrt{\frac{k}{m}}$$

to get

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

ii) Solve the initial value problem $x(0) = 0.2$, $x'(0) = 0$.
The general solution is $x(t) = A\sin(\omega t) + B\cos(\omega t)$ and $x'(0) = 0$ means $A = 0$ and so $x(0) = 0.2$ gives $B = 0.2$.
iii) Solve the initial value problem $x(0) = 0.0$, $x'(0) = 0.2$. Comment on the difference between the two problems.
Arguing as in the last problem we get $A = 0.2/\omega$ and $B = 0$. The difference comes from the fact that differentiation not only turns cosines to negative sines and sines to cosines, but also brings out an ω term.
iv) Multiply the equation of motion by $x'(t)$ and integrate with respect to t. You will need to use the Chain Rule.
The equation of motion multiplied by $x'(t)$ reads

$$mx'(t)x''(t) + kx(t)x'(t) = 0$$

now notice that using the Chain Rule

$$\frac{d}{dt}\frac{x(t)^2}{2} = x(t)x'(t)$$

and

$$\frac{d}{dt}\frac{x'(t)^2}{2} = x'(t)x''(t).$$

Thus we can integrate our modified equation of motion to get

$$\frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2 = C$$

v) In the expression you found in iv identify the potential and kinetic energy terms and thus argue that you have derived the conservation of energy.
Since $x'(t) = v(t)$ the first term is the kinetic energy, the second term is thus the potential energy stored in the stretched string.

vi) Now say the mass experiences some external forcing, which we will label $F_e(t)$. Show that the same governing equation still applies but the right hand side is now non-zero.
The balance of forces now gives

$$m\frac{d^2x}{dt^2} = -kx + F_e(t)$$

rearranging and multiplying by $x'(t)$ gives

$$mx'(t)x''(t) + kx(t)x'(t) = F_e(t)x'(t)$$

vii) Rederive the conservation of energy equation and explain how the total energy changes.
Upon integrating we get

$$\frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2 = \int F_e(t)x'(t)dt$$

(Q6) Consider the curve given by $y = x^{3/2}$ on the interval $[1, 2]$.
i) Find the arclength using the formula from your Course Materials book.
You will note that this problem is essentially example 1 on page 17. The integral you have to do is

$$L = \int_1^2 \sqrt{1 + \frac{9}{4}x}dx$$

The antiderivative is found in example 1 and the resulting algebra is a big mess. In the end you should find that to three decimal places $L = 2.086$.

ii) Sketch the curve. Now divide the interval into 5 sub-intervals and estimate the arclength by using line segments to approximate the curve. How good is the approximation.
Again the sketch is in example 1 on page 17, so I won't reproduce it. What I am really asking you to do is the trapezoid rule for numerical integration, though you can just do it *ad hoc*. First we find the 6 points needed for the line segments (I keep three decimal places): $(1, 1)$, $(1.2, 1.315)$, $(1.4, 1.657)$,$(1.6, 2.024)$,$(1.8, 2.415)$,$(2.0, 2.282)$. Then you find the length of each segment and add them up. You can do this by hand with a calculator, but I wrote a little loop for this in Maple:

```
N=5; dx:=1/N; L:=0;
for i from 1 to N do;
  x2:=1+dx*i; x1:=1+dx*(i-1);
  y2:=evalf(subs(x=x2,g)); y1:=evalf(subs(x=x1,g));
  L:=L+sqrt((y2-y1)^2+(x2-x1)^2);
end do;
```

When I execute this I get, to three decimal places 2.086, which matches the exact result. If you kept more decimal places you would see that the approximation is a bit off. Of course making N bigger uses more line segments and so you expect a better approximation. Still it may be a surprise that even with 5 line segments the approximation is so good.