Math 138 Physics Based Section Assignment 4

(Q1) i) Read the Newton's law of Temperature Change on page 35 in the course materials (hand nothing in).
ii) Rewrite example 5 in your own words.
iii) Do Exercise 3 on page 35.

(Q2) Consider a container of sand attached to a spring. The container moves over a frictionless surface, and the force due to the spring is given by $F = -kx(t)$. However the box is leaking sand so that its mass is a function of time $m(t)$.
i) Derive the governing equation

$$\frac{d^2x}{dt^2} + \frac{k}{m(t)}x(t) = 0$$

ii) This problem cannot be generally solved in closed form (as a formula). However if $m(t)$ changes slowly you can imagine that the solution is going to be a lot like the solution when m is constant. What happens to the frequency of oscillations as $m(t)$ decreases? Sketch a sample solution.

(Q3) Consider the Maxwell model of a spring (with constant k) and dashpot (with constant D) hooked up in series. The governing equation relating the position $x(t)$ and force $f(t)$ is given by

$$\frac{dx(t)}{dt} = \frac{1}{k}\frac{df(t)}{dt} + \frac{1}{D}f(t)$$

(we derived this in class)
i) If $r(0) = 0$ and $f(t) = H(t)$ use Laplace transforms to solve the differential equation for $r(t)$ and sketch the solution when $D = 1$ and $k = 2$.
ii) If $f(0) = 0$ and $x(t) = H(t)$ use Laplace transforms to solve the differential equation for $f(t)$ and sketch the solution when $D = 1$ and $k = 2$.
iii) If the "relaxation time" is defined as the time typical of change in a given physical situation. Define, with an explanation a relaxation time for each of the two solutions you found above.

(Q4) Solve the following linear DEs using the method of integrating factors and sketch the solution:
i)

$$x'(t) - x(t) = 1 - t$$

with initial condition $x(0) = 0$.
ii)

$$x'(t) - x(t) = 1 - t$$

with initial condition $x(0) = 3$.
iii)
$$x'(t) - 2tx(t) = 1 - 2t^2$$
with initial condition $x(0) = 0$.
iv)
$$x'(t) - 2tx(t) = 1 - 2t^2$$
with initial condition $x(0) = 2$.
v) Use Maple for parts v and vi.
$$x'(t) + \cos(t)x(t) = \frac{\cos(t)}{t} - \frac{1}{t^2}$$
with initial condition $x(2) = 0$. Why can we not give a condition at $t = 0$?
vi)
$$x'(t) + \cos(t)x(t) = \frac{\cos(t)}{t} - \frac{1}{t^2}$$
with initial condition $x(2) = -1$.

(Q5) Up to now we have used the Laplace transform to solve a single differential equation. It is just as useful for systems of equations.
i) Let's start with something we know. Consider the simplest SHO equation:
$$\frac{d^2x}{dt^2} + x = 0$$
and rewrite it as a system for the two variables $x(t)$ and $v(t)$.
This is the only part I did by hand, the remainder can safely be done with Maple (I have posted an example worksheet on how to do it).
ii) Take the Laplace transform of the system and use the initial conditions $x(0) = 2$ and $v(0) = -1$ to solve for $X(s)$ and $V(s)$. Finally find $x(t)$ and $v(t)$.
iii) Solve the system
$$\begin{aligned} x'(t) &= 3x(t) + v(t) \\ v'(t) &= x(t) + v(t) \end{aligned} \tag{1}$$
with the initial conditions $x(0) = 1$ and $v(0) = -1$.
iv) Solve the system
$$\begin{aligned} x'(t) &= 3x(t) + v(t) \\ v'(t) &= x(t) + v(t) \end{aligned} \tag{2}$$
with the initial conditions $x(0) = 1$ and $v(0) = -10$.

(Q6) Consider a point particle moving under the action of gravity (take the gravitational acceleration to be $g = 10$ m s^2. Formulate and solve the problem with the particle initially

found at the origin.
i) Assume there is no wind and the initial velocity is given by $(3, 4)$. Find the time at which the particle hits the ground and the horizontal distance travelled.
ii) Now assume that there is a constant horizontal wind $(-10, 0)$, how does the distance travelled change? Does the time in the air change?
iii) If the wind is given by $(v_w, 0)$ find v_w so that the particle lands exactly where it started and sketch the trajectory.

(Q7) Given the position vector sketch the path in each case and indicate the direction of travel.
i) $\vec{x} = (\cos(t), \sin(t))$
ii) $\vec{x} = (\cos(2t), \sin(2t))$
iii) $\vec{x} = (\cos(t), 3\sin(t))$
iv) $\vec{x} = (\cos(t), -3\sin(t))$
v) $\vec{x} = (3\cos(t), \sin(t))$
vi) $\vec{x} = (\cos(t), \sin(2t))$

(Q8)i) If we have two particles that start at the same position $x(0) = 0$ with velocities $v_1(t)$ and $v_2(t)$. Find a condition that will guarantee that $x_1(t) > x_2(t)$ for all $t > 0$.
ii) If $x'(t) = 1/\sqrt{1-t}$ and $x(0) = 0$ make a careful argument to find $x(1)$.
iii) Would the same argument work for $x'(t) = (1-t)^{-1}$? Explain the difference between the two cases.
iv) Use your result in part ii) to interpret the differential equation

$$\frac{dx}{dt} = \frac{1}{\sqrt{|1-t|}}$$

with $x(0) = 0$ and sketch a solution for $0 \leq t \leq 3$. HINT: you will need to make an assumption about what happens in a tiny interval around $t = 1$.