MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 1

Due September ~~20~~ at the beginning of class
postponed to Fri. Sept. 22/06

Evaluate the following integrals **using the method indicated**.
Note: on a test or exam you will not typically be given the method of integration to use.

Justify each answer, showing all necessary steps. Simplify your answers as much as possible (without a calculator, of course).

1. Use u–substitution (also referred to as "change of variable"):
   (a) $\int \frac{(\ell n\, x)^2}{x} dx$
   (b) $\int_0^1 x^3 e^{1-x^4}\, dx$
   (c) $\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+\cos t} dt$

2. Use integration by parts:
   (a) $\int e^{-\theta} \sin 3\theta \, d\theta$
   (b) $\int_0^1 s \tan^{-1} s \, ds$

3. Use a suitable trigonometric substitution:
   (a) $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} dx$
   (b) $\int \frac{dt}{(1+t^2)^{3/2}}$

4. Use partial fractions:
   (a) $\int \frac{10x+20}{(x^2-1)(x^2+4)} dx$
   (b) $\int_1^4 \frac{2p^2+3}{p(p+1)^2} dp$

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Fall 2006 — Assignment 1 — SOLUTIONS

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1. a) $I = \int \frac{(\ln x)^2}{x}\, dx \qquad u = \ln x$ (1) $\Rightarrow du = \frac{dx}{x}$ (1)

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$$\Rightarrow I = \int u^2\, du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C}$$

(1) (1) (1)

b) $I = \int_0^1 x^3 e^{1-x^4}\, dx$

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Solution 1: $I = e\int_0^1 x^3 e^{-x^4}\, dx$

let $u = -x^4$ (1) $\Rightarrow du = -4x^3 dx \Rightarrow -\frac{du}{4} = x^3 dx$, (1)

$\boxed{x = 0 \Rightarrow u = 0, \quad x = 1 \Rightarrow u = -1}$ — (2)

$$\Rightarrow I = e\int_0^{-1}\left(-\frac{du}{4}\right)e^u = \boxed{-\frac{e}{4}\int_0^{-1} e^u\, du}$$ — (1)

$$= \boxed{\frac{e}{4}\int_{-1}^{0} e^u\, du} = \frac{e}{4}\left[e^u\right]_{-1}^{0}$$ (1) (1)

Note the reversal of integration limits requires multiplying the integral by -1. (See page 361, text.)

$$= \frac{e}{4}\left(e^0 - e^{-1}\right) = \frac{e}{4}\left(1 - e^{-1}\right)$$

$$= \boxed{\frac{e}{4} - \frac{1}{4}} \overset{OR}{=} \boxed{\frac{1}{4}(e-1)}$$ (1)

OR Solution 2): $I = \int_0^1 x^3 e^{1-x^4}\, dx, \quad u = 1 - x^4,$ (1) $du = -4x^3 dx$

(1) $\Rightarrow \boxed{x^3 dx = -\frac{du}{4}}, \quad \left\{\begin{array}{l} x = 0 \Rightarrow u = 1 \\ x = 1 \Rightarrow u = 0 \end{array}\right.$ — (2)

$$\Rightarrow I = \int_1^0 \left(-\frac{du}{4}\right)e^u = \frac{1}{4}\int_0^1 e^u\, du = \frac{1}{4}\left[e^u\right]_0^1 = \boxed{\frac{1}{4}(e-1)}$$ — (1)

(1) (1) (1)

1c) $I = \int_0^{\pi/2} \frac{\sin t}{1+\cos t}\, dt$ $\quad u = 1+\cos t$ (1)

$du = -\sin t\, dt$

or $-du = \sin t\, dt$ (1)

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(2) $\begin{cases} t=0 \Rightarrow u = 1+\cos 0 = 1+1 = 2, \\ t=\frac{\pi}{2} \Rightarrow u = 1+\cos\frac{\pi}{2} = 1+0 = 1 \end{cases}$

$\Rightarrow I = \int_2^1 \frac{-du}{u} = \int_1^2 \frac{du}{u} = \left[\ln u\right]_1^2$ (1) (1) (1)

$= \ln 2 - \underbrace{\ln 1}_{=0} = \boxed{\ln 2}$ (1)

2a) $I = \int e^{-\theta} \sin 3\theta\, d\theta$ $\quad$ IBP:*

$u = \sin 3\theta \quad v = -e^{-\theta}$

$du = 3\cos 3\theta\, d\theta \quad dv = e^{-\theta}\, d\theta$ (2)

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$\Rightarrow I = -e^{-\theta}\sin 3\theta - \int -e^{-\theta}(3\cos 3\theta\, d\theta) \leftarrow$ (2)

$= -e^{-\theta}\sin 3\theta + 3\int e^{-\theta}\cos 3\theta\, d\theta$ $\quad$ IBP again:*

$u = \cos 3\theta \quad v = -e^{-\theta}$

$du = -3\sin 3\theta\, d\theta \quad dv = e^{-\theta}\, d\theta$ (2)

$\Rightarrow I = -e^{-\theta}\sin 3\theta + 3\left[-e^{-\theta}\cos 3\theta - \int -e^{-\theta}(-3\sin 3\theta\, d\theta)\right] \leftarrow$ (2)

$\Rightarrow I = -e^{-\theta}\sin 3\theta - 3e^{-\theta}\cos 3\theta - 9\underbrace{\int e^{-\theta}\sin 3\theta\, d\theta}_{= I\text{, the given integral}}$

(1)

$\Rightarrow \boxed{10I = -e^{-\theta}\sin 3\theta - 3e^{-\theta}\cos 3\theta}$

$\Rightarrow \boxed{I = -\frac{1}{10}e^{-\theta}\sin 3\theta - \frac{3}{10}e^{-\theta}\cos 3\theta + C}$ (2) the C is ESSENTIAL!

*NOTE: Could let $u = e^{-\theta}$ (both times) instead.

2b) $I = \int_0^1 s \tan^{-1} s \, ds$ $\quad$ $\underline{IBP}$: $u = \tan^{-1} s$ $\quad v = \frac{s^2}{2}$ $\quad$ (2)

$*\, du = \frac{ds}{1+s^2}$ $\quad dv = s\,ds$

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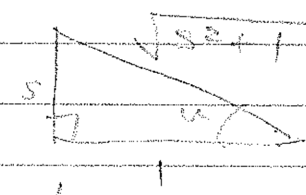
* Aside

$u = \tan^{-1} s \Rightarrow \tan u = s \Rightarrow$



where

$\tan(u(s)) = s \Rightarrow \frac{d}{ds}(\tan u(s)) = \frac{d}{ds}(s)$

$\Rightarrow \frac{d}{ds}\left(\frac{\sin u(s)}{\cos u(s)}\right) = 1$

Need quotient and chain rules

$\Rightarrow \frac{\left(\cos(u(s))\frac{du}{ds}\right)\cos(u(s)) - \sin(u(s))\left(-\sin(u(s))\frac{du}{ds}\right)}{\cos^2(u(s))} = 1$

$\Rightarrow \frac{du}{ds}\frac{(\cos^2(u(s)) + \sin^2(u(s)))}{\cos^2(u(s))} = 1$

$\Rightarrow \frac{du}{ds}\frac{1}{\cos^2(u(s))} = 1$

Triangle $\Rightarrow$ $\cos u(s) = \frac{1}{\sqrt{s^2+1}}$

$\Rightarrow \frac{du}{ds} \cdot \frac{1}{s^2+1} = 1$ $\quad \therefore \boxed{du = \frac{ds}{s^2+1}}$ *

$\Rightarrow I = \left[\frac{s^2}{2}\tan^{-1} s\right]_0^1 - \int_0^1 \frac{s^2}{2}\frac{ds}{1+s^2}$ $\quad$ (3) (1 for s, 1 lim's)

$\frac{s^2}{1+s^2}$: LONG DIVISION

$$\begin{array}{r} 1 \\ 1+s^2 \overline{) s^2 + 0s + 0} \\ \underline{s^2 + 1} \\ -1 \end{array}$$

$\therefore \frac{s^2}{1+s^2} = 1 - \frac{1}{s^2+1}$

$= \left(\frac{1}{2}\tan^{-1} 1 - 0\right) - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{s^2+1}\right) ds$ $\quad$ (1)

(1) $= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left[s - \tan^{-1} s\right]_0^1$ $\quad$ (1)

or

$= \boxed{\frac{\pi}{8}} - \frac{1}{2}\left(1 - \tan^{-1} 1 - (0 - \tan^{-1} 0)\right)$

(1) $= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4} - \frac{1}{2}}$ $\quad$ (1)

3a) $I = \int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{4^2-x^2}}$

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let $x = 4\sin\theta$
$\Rightarrow dx = 4\cos\theta\,d\theta$ } 2

$\begin{cases} x = 0 = 4\sin\theta \Rightarrow \theta = 0, \\ x = 2\sqrt{3} = 4\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3} \end{cases}$ 2

$\Rightarrow I = \int_0^{\pi/3} \frac{(64\sin^3\theta)\,4\cos\theta\,d\theta}{\sqrt{16-(4\sin\theta)^2}}$

$= 256 \int_0^{\pi/3} \frac{\sin^3\theta\cos\theta}{\sqrt{16(1-\sin^2\theta)}}\,d\theta$ ← 1

$= \frac{256}{4} \int_0^{\pi/3} \frac{\sin^3\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta$ 1

$= \boxed{64 \int_0^{\pi/3} \sin^3\theta\,d\theta}$ EXPAND $\sin^3\theta$

1 $= 64 \int_0^{\pi/3} \sin^2\theta \sin\theta\,d\theta$ $\cos^2\theta + \sin^2\theta = 1$

1 $= 64 \int_0^{\pi/3} (1-\cos^2\theta)\sin\theta\,d\theta$

2
let $u = \cos\theta$
$du = -\sin\theta\,d\theta$,
$\begin{cases} \theta = 0 \Rightarrow u = \cos 0 = 1, \\ \theta = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2} \end{cases}$

1 $= 64 \int_1^{1/2} -(1-u^2)\,du$

1 $= -64 \int_{1/2}^{1} (-1+u^2)\,du$

1 $= -64 \left[-u + \frac{u^3}{3}\right]_{1/2}^{1}$

$= -64\left(-1+\frac{1}{3} - \left(-\frac{1}{2}+\frac{1}{24}\right)\right) = 64\left(\frac{24-8-12+1}{24}\right)$

$\therefore \boxed{I = \frac{40}{3}}$ 1

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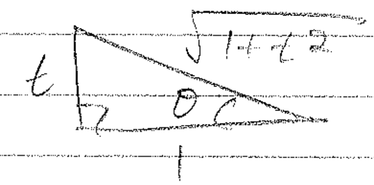
3b) $I = \int \frac{dt}{(1+t^2)^{3/2}}$ $\quad$ let $\left.\begin{aligned} t &= \tan\theta \\ dt &= \sec^2\theta\, d\theta \end{aligned}\right\}$ 2

$\Rightarrow I = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}} = \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$ 2

$= \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta\, d\theta} = \int \frac{d\theta}{\sec\theta}$ 1

$= \int \cos\theta\, d\theta = \boxed{\sin\theta + C}$ — 2

1 — * $\left\{ \begin{aligned} &\text{Since } t = \tan\theta \Rightarrow \\ &\Rightarrow \sin\theta = \frac{t}{\sqrt{1+t^2}} \end{aligned} \right.$



$\therefore \boxed{I = \frac{t}{\sqrt{1+t^2}} + C}$ — 1 **

Note: Since $t = \tan\theta \Rightarrow \theta = \tan^{-1} t$.
Recall: $-\frac{\pi}{2} < \tan^{-1} t < \frac{\pi}{2}$.

So, $\boxed{\sin\theta = \sin(\tan^{-1} t)}$ is defined for all $t \in \mathbb{R}$ (i.e., there are no restrictions on $\sin(\tan^{-1} t)$ that affect our result).

$\therefore \boxed{I = \sin(\tan^{-1} t) + C}$ **
is also correct
(and equivalent to the answer above).

4a) $I = \int \frac{10x+20}{(x^2-1)(x^2+4)}\,dx = \int \frac{10x+20}{(x+1)(x-1)(x^2+4)}\,dx$

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$= \int \left(\frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4} \right) dx \leftarrow$ 3

3 $\longrightarrow$ $= \int \frac{A(x-1)(x^2+4) + B(x+1)(x^2+4) + (Cx+D)(x^2-1)}{(x+1)(x-1)(x^2+4)}\,dx$

Since denominators are equal, equate numerators:

1 $\Rightarrow 10x+20 = A(x-1)(x^2+4) + B(x+1)(x^2+4) + (Cx+D)(x^2-1)$

$\Rightarrow 10x+20 = A(x^3+4x-x^2-4) + B(x^3+4x+x^2+4) + Cx^3 - Cx + Dx^2 - D$

$\Rightarrow 10x+20 = x^3(A+B+C) + x^2(-A+B+D) + x(4A+4B-C) + (-4A+4B-D)$

Match coefficients on LHS to RHS:

$$\left.\begin{aligned} x^3&: 0 = A+B+C && (1)\\ x^2&: 0 = -A+B+D && (2)\\ x^1&: 10 = 4A+4B-C && (3)\\ x^0&: 20 = -4A+4B-D && (4) \end{aligned}\right\} \text{4}$$

4 - steps
4 - A, B, C, D

- Add Eq's (2) & (4) $\Rightarrow 20 = -5A+5B$ (5)
- Add Eq's (1) & (3) $\Rightarrow 10 = 5A+5B$ (6)
- Add Eq's (5) & (6) $\Rightarrow 30 = 10B \Rightarrow \boxed{B=3}$
- From Eq (5) (or could use Eq (6)) $\Rightarrow 20 = -5A+15 \Rightarrow \boxed{A=-1}$
- From Eq (1) $\Rightarrow 0 = -1+3+C \Rightarrow \boxed{C=-2}$
- From Eq (2) (or could use Eq (3)) $\Rightarrow 0 = 1+3+D \Rightarrow \boxed{D=-4}$

- see next page -

4a) cont'd

$$\Rightarrow I = \int \left(\frac{-1}{x+1} + \frac{3}{x-1} + \frac{-2x-4}{x^2+4} \right) dx \quad (1)$$

$$= -\ln|x+1| + 3\ln|x-1| - \int \left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} \right) dx$$

use substitution $u = x^2+4$, OR, "see directly" that the antiderivative is $\ln(x^2+4)$ (since $\frac{d}{dx}\ln(x^2+4) = \frac{2x}{x^2+4}$)

Let $x = 2\tan\theta$
$\Rightarrow dx = 2\sec^2\theta\, d\theta$
etc., OR, recall

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)} = \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right),$$

So, $$\int \frac{4\,dx}{x^2+4} = 4\int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta+4}$$

$$= \frac{8}{4}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1}, \quad (\tan^2\theta+1=\sec^2\theta)$$

$$= 2\int d\theta = 2\theta + C, \quad \left\{ \begin{array}{l} x = 2\tan\theta \\ \Rightarrow \frac{x}{2} = \tan\theta \\ \Rightarrow \theta = \tan^{-1}\frac{x}{2} \end{array} \right.$$

$$= 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\therefore \boxed{I = -\ln|x+1| + 3\ln|x-1| - \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

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4b) $I = \int_1^4 \frac{2p^2+3}{p(p+1)^2}\,dp$

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$$\frac{2p^2+3}{p(p+1)^2} = \underbrace{\frac{A}{p} + \frac{B}{p+1} + \frac{C}{(p+1)^2}}_{3} = \underbrace{\frac{A(p+1)^2 + Bp(p+1) + Cp}{p(p+1)^2}}_{3}$$

$$\Rightarrow 2p^2+3 = A(p^2+2p+1) + Bp^2 + Bp + Cp \quad \leftarrow 1$$

$$\Rightarrow \begin{cases} p^2: \ 2 = A + B \\ p^1: \ 0 = 2A + B + C \\ p^0: \ \boxed{3 = A} \end{cases} \Rightarrow 2 = 3 + B \Rightarrow \boxed{B = -1}$$

p^1 coeff.: $\Rightarrow 0 = 2(3) + (-1) + C \Rightarrow \boxed{C = -5}$

2 - work
3 - A,B,C

$$\Rightarrow I = \int_1^4 \left(\frac{3}{p} - \frac{1}{p+1} - \frac{5}{(p+1)^2}\right) dp \quad 1$$

$$= \left[3\ln|p| - \ln|p+1| + \frac{5}{p+1}\right]_1^4 \quad 3$$

$$= 3\ln 4 - \ln 5 + \frac{5}{5} - \left(3\cancel{\ln 1}_{0} - \ln 2 + \frac{5}{2}\right)$$

$$= 3\ln 4 - \ln 5 + 1 + \ln 2 - \frac{5}{2}$$

$$\therefore \boxed{I = 3\ln 4 - \ln 5 + \ln 2 - \frac{3}{2}} \text{ — } 4$$

OR, since

$\ln 4 = \ln 2^2 = 2\ln 2$,

$$\left.\begin{aligned} \Rightarrow I &= 3(2\ln 2) - \ln 5 + \ln 2 - \tfrac{3}{2} \\ &= 7\ln 2 - \ln 5 - \tfrac{3}{2} \\ &= \ln\left(\tfrac{2^7}{5}\right) - \tfrac{3}{2} \end{aligned}\right\} \text{optional.}$$