Math 138 Assignment #5

Q1) I enclose the plots separately. Below I give comments on why I asked the various problems

i) Here $x = y^2$ or $y = \sqrt{x}$ since $t > 0$ we have the top half of a parabola opening around the x-axis

ii) Here $y = x$ so we are on a straight line. Note $\vec{v} = 2(t,t)$ When $t < 0$ we are heading towards the origin and when $t > 0$ we are heading away from the origin. At $t = 0$ we have $\vec{v} = (0,0)$ so the particle on the curve stops.

iii) The usual circle, once around.

iv) The usual circle twice around

v) Ellipse $\left(\frac{x}{2}\right)^2 + y^2 = 1$, one trip around.

vi) This is a trick as the tangent lines are also along the line $y = x$. At $t=0$ $\vec{v} = (0,0)$ and the particle stops.

(Q4) ii) $$\frac{d\ell}{d\alpha} = \int_{t=0}^{t=5} \sqrt{4t^2 + 9t^4}\, dt$$

but the arclength ≥ 0 by definition so $\frac{d\ell}{d\alpha} > 0$ and ℓ is an

increasing function of α

(Q5 i) $\vec{x}(t) = (\alpha t + \cos t, \sin t)$

$$\frac{dx}{dt} = \alpha - \sin t \; ; \; \frac{dy}{dt} = \cos t$$

$$\ell = \int_0^T \sqrt{(\alpha - \sin t)^2 + \cos^2 t}\; dt$$

ii) see Maple worksheet

iii) when $\alpha < 1$ increasing $\alpha \Rightarrow$ smaller ℓ

$\alpha > 1$ increasing $\alpha \Rightarrow$ larger ℓ

Q3 cont'd) To find T notice that in one revolution of the wheel, the bike travels $2\pi R$ meters to the right. Here $R = 0.5$m so one revolution is π meters

But we know that the horizontal velocity of the bike is 2 m/s

one revolution $= \pi = T \cdot 2\,m/s$

and thus $\boxed{T = \frac{\pi}{2} \text{ seconds}}$

Notice a faster horizontal velocity will mean a smaller period of revolution.

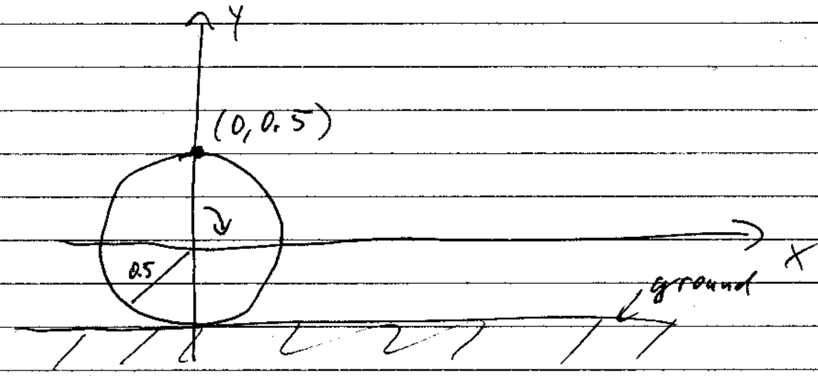
Q4 i) $(x(t), y(t)) = \alpha\,(t^2, t^3) \quad 0 \le t \le 5$

$$\left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = \alpha\,(2t, 3t^2)$$

$$l = \int_{t=0}^{t=5} \sqrt{\alpha^2 4t^2 + \alpha^2 9t^4}\; dt$$

$$= \alpha \int_{t=0}^{t=5} \sqrt{4t^2 + 9t^4}\; dt$$

(Q3) place the origin at the center of the wheel at $t=0$



The wheel spins clockwise as the bike moves.
First, say the wheel spins in place.
From the diagram we see that

$$\vec{x}(t) = \tfrac{1}{2}\left(\sin\left(\frac{2\pi t}{T}\right), \cos\left(\frac{2\pi t}{T}\right)\right)$$

Here T is the time to complete one revolution.

When the wheel moves the above spinning motion must have superimposed on it a translation in the x-direction

$$\vec{x}(t) = \left[2t + \tfrac{1}{2}\sin\left(\frac{2\pi t}{T}\right), \tfrac{1}{2}\cos\left(\frac{2\pi t}{T}\right)\right]$$

How can we find T?

Q6 Note that $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

so $\text{proj}_{\vec{v}} \vec{u} = \vec{u} \cdot \hat{v}$

$\vec{u}$	$\vec{v}$	$\hat{v}$	$\text{proj}_{\vec{v}} \vec{u}$
$(3,4)$	$(3,4)$	$\frac{1}{5}(3,4)$	1
$(1,0)$	$(3,4)$	$\frac{1}{5}(3,4)$	$3/5$
$(0,1)$	$(3,4)$	$\frac{1}{5}(3,4)$	$4/5$
$(3,4)$	$(0,1)$	$(0,1)$	4
$(3,4)$	$(-4,3)$	$\frac{1}{5}(-4,3)$	0

Q7 A circle centered at the origin has the equation

$$x(t)^2 + y(t)^2 = R^2$$

take derivative of both sides with respect to t

$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

or dividing by 2 & using the definition of dot product

$$(x(t), y(t)) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0$$

So now in reverse start with

$$\vec{x}'(t) \cdot \vec{x}(t) = 0 \quad \text{for all } t$$

Q7 continued write it out as

$$x(t)\frac{dx}{dt} + y(t)\frac{dy}{dt} = 0$$

multiply by 2 and integrate with respect to "t"

$$\int\left(2x(t)\frac{dx}{dt}\right)dt + \int\left(2y(t)\frac{dy}{dt}\right)dt = C$$

get

$$x(t)^2 + y(t)^2 = C$$

finally set $C = R^2$

Q8 iv) Since $\vec{p}(t) = (x(t), v(t))$ then

$$\vec{p}\,'(t) = \left(\frac{dx}{dt}, \frac{dv}{dt}\right)$$

$$= (v(t), a(t))$$ $a(t) \equiv$ acceleration

and only $a(t)$ is new.

(Q9) i) $\vec{F} \cdot \vec{x}'(t) = (1,1) \cdot (2t, 3t^2)$

$$\text{Work} = \int_{t=0}^{t=1} (2t + 3t^2)\,dt$$

$$= t^2 + t^3 \Big|_0^1 = 2$$

ii) $\vec{F} \cdot \vec{x}'(t) = (a,b) \cdot (2t, 3t^2)$

$$\text{Work} = \int_{t=0}^{t=1} (2at + 3bt^2)\,dt$$

$$= (at^2 + bt^3)\Big|_0^1 = a + b$$

iii) $\vec{F} \cdot \vec{x}'(t) = (t^2, t^3) \cdot (2t, 3t^2)$

$$= 2t^3 + 3t^5$$

$$\text{Work} = \int_{t=0}^{t=1} (2t^3 + 3t^5)\,dt$$

$$= \left(\frac{2}{4}t^4 + \frac{3}{6}t^6\right)\Big|_0^1$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1$$