MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 11

1: Find the first partial derivatives for each of the following functions at the given point.

(a) $f(x,y)=\tan^{-1}(x\sqrt{y})$ at $(1,4)$.

Solution: We have $\frac{\partial f}{\partial x}=\frac{\sqrt{y}}{1+x^2y}$ so $\frac{\partial f}{\partial x}(1,4)=\frac{2}{5}$, and $\frac{\partial f}{\partial y}=\frac{x/2\sqrt{y}}{1+x^2y}$ so $\frac{\partial f}{\partial y}(1,4)=\frac{1/4}{5}=\frac{1}{20}$.

(b) $g(x,y,z)=x^{y/z}$ at $(2,-2,1)$.

Solution: We have $\frac{\partial g}{\partial x}=\frac{y}{z}\cdot x^{\frac{y}{z}-1}$ so $\frac{\partial g}{\partial x}(2,-2,1)=-2\cdot 2^{-3}=-\frac{1}{4}$, and we have $\frac{\partial g}{\partial y}=\ln x\cdot x^{y/z}\cdot\frac{1}{z}$ so $\frac{\partial g}{\partial y}(2,-2,1)=\ln 2\cdot 2^{-2}=\frac{1}{4}\ln 2$, and we have $\frac{\partial g}{\partial z}=\ln x\cdot x^{y/z}\cdot\frac{-y}{z^2}$ so $\frac{\partial g}{\partial z}(2,-2,1)=\ln 2\cdot 2^{-2}\cdot 2=\frac{1}{2}\ln 2$.

2: Let $u(x,t)=\dfrac{1}{\sqrt{t}\,e^{x^2/4t}}$. Show that $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ (this equation is called the *heat equation*).

Solution: Note that we can write $u=t^{-1/2}\,e^{-x^2/4t}$. We have $\frac{\partial u}{\partial t}=\left(-\frac{1}{2}\,t^{-3/2}+t^{-1/2}\,\frac{x^2}{4t^2}\right)e^{-x^2/4t}$, and we have $\frac{\partial u}{\partial x}=t^{-1/2}\cdot\frac{-2x}{4t}\cdot e^{-x^2/4t}=-\frac{1}{2}\,t^{-3/2}x\,e^{-x^2/4t}$ and so $\frac{\partial^2 u}{\partial x^2}=-\frac{1}{2}\,t^{-3/2}\left(1-\frac{2x^2}{4t}\right)e^{-x^2/4t}=\left(-\frac{1}{2}\,t^{-3/2}+\frac{x^2}{4}\,t^{-5/2}\right)e^{-x^2/4}=\frac{\partial u}{\partial t}$.

3: Let P be the tangent plane to the surface $z=\sqrt{x+e^{-xy}}$ at the point $(3,0,2)$. Find the standard equation for P, and find a vector (or parametric) equation for the line of intersection of P with the xy-plane.

Solution: We have $z=\sqrt{x+e^{-xy}}$ so $z(3,0)=2$, and we have $\frac{\partial z}{\partial x}=\frac{1-y\,e^{-xy}}{2\sqrt{x+e^{-xy}}}$ so $\frac{\partial z}{\partial x}(3,0)=\frac{1}{4}$, and we have $\frac{\partial z}{\partial y}=\frac{-x\,e^{-xy}}{2\sqrt{x+e^{-xy}}}$ so $\frac{\partial z}{\partial y}(3,0)=-\frac{3}{4}$. Thus the equation of the tangent plane at $(3,0,2)$ is $z=2+\frac{1}{4}(x-3)-\frac{3}{4}\,y$, that is $4z=8+x-3-3y$, or equivalently $x-3y-4z=-5$. To find the intersection with the xy-plane, we put in $z=0$ to get $x-3y=-5$. A vector equation for this line is $(x,y)=(1,2)+t(3,1)$.

4: Find all the points at which the surface $z=2x^3+xy^2+5x^2+y^2$ has a horizontal tangent plane.

Solution: We have $\frac{\partial z}{\partial x}=6x^2+y^2+10x$ and $\frac{\partial z}{\partial y}=2xy+2y$. The tangent plane will be horizontal when both $\frac{\partial z}{\partial x}=0$ and $\frac{\partial z}{\partial y}=0$. Note that $\frac{\partial z}{\partial y}=0\iff 2y(x+1)=0\iff x=-1$ or $y=0$. When $x=-1$, we have $\frac{\partial z}{\partial x}=y^2-4=(y+2)(y-2)$ so $\frac{\partial z}{\partial x}=0\iff y=\pm 2$. When $y=0$ we have $\frac{\partial z}{\partial x}=6x^2+10x=6x\left(x+\frac{5}{3}\right)$ so $\frac{\partial z}{\partial x}=0\iff x=0$ or $x=-\frac{5}{3}$. Thus the tangent plane is horizontal at the points where $(x,y)=(-1,2)$, $(-1,-2)$, $(0,0)$ and $\left(-\frac{5}{3},0\right)$, that is at the points $(x,y,z)=(-1,2,3)$, $(-1,-2,3)$, $(0,0,0)$ and $\left(-\frac{5}{3},0,\frac{125}{27}\right)$.

5: Redo problem 5 from last week's assignment in the following way. Let $p=(1,-1,\sqrt{2})$.

(a) Find the equation of the tangent plane to the cone $z=\sqrt{x^2+y^2}$ at the point p.

Solution: For $z=\sqrt{x^2+y^2}$, we have $\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$, so $z(1,-1)=\sqrt{2}$, $\frac{\partial z}{\partial x}(1,-1)=\frac{1}{\sqrt{2}}$ and $\frac{\partial z}{\partial y}(1,-1)=-\frac{1}{\sqrt{2}}$. The equation of the tangent plane at $(1,-1,\sqrt{2})$ is $z=\sqrt{2}+\frac{1}{\sqrt{2}}(x-1)-\frac{1}{\sqrt{2}}(y+1)$, which we can write as $\sqrt{2}z=2+(x-1)-(y+1)$, or as $x-y-\sqrt{2}\,z=0$.

(b) Find the equation of the tangent plane to the sphere $x^2+y^2+z^2=4x$ at the point p.

Solution: For $z=\sqrt{4x-x^2-y^2}$ we have $\frac{\partial z}{\partial x}=\frac{2-x}{\sqrt{4x-x^2-y^2}}$ and $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{4x-x^2-y^2}}$ and so $z(1,-1)=\sqrt{2}$, $\frac{\partial z}{\partial x}(1,-1)=\frac{1}{\sqrt{2}}$ and $\frac{\partial z}{\partial y}(1,-1)=\frac{1}{\sqrt{2}}$. Thus the equation of the tangent plane is $z=\sqrt{2}+\frac{1}{\sqrt{2}}(x-1)+\frac{1}{\sqrt{2}}(y+1)$, or equivalently $\sqrt{2}z=2+(x-1)+(y+1)$, or $x+y-\sqrt{2}\,z=-2$.

(c) Find a vector (or parametric) equation for the line of intersection of the above two planes.

Solution: We solve the two equations $x-y-\sqrt{2}\,z=0$ (1) and $x+y-\sqrt{2}\,z=-2$ (2). Subtract (1) from (2) to get $2y=-2$ so $y=-1$. Put $y=-1$ into equation (1) to get $x-\sqrt{2}\,z=-1$, so $x=-1+\sqrt{2}\,z$. If we set $z=t$ then we have $x=-1+\sqrt{2}\,t$ and $y=-1$, so a vector equation for the line of intersection is $(x,y,z)=(-1+\sqrt{2}\,t,-1,t)$. (Note that this is not identical to the solution we obtained in assignment 10; it is an alternate vector equation for the same line).