MATH 136 Solutions Assignment 3 Fall/05

October 6, 2005

Following are the solutions. Most of the solutions are given using MATLAB. However, you are not required to use MATLAB for these assignments.

Contents

1 page 48 #24

12 marks

Solution.

(a) **TRUE**
Justification: Matrix-vector multiplication Ax is equivalent to the linear combination $\sum_{j=1}^{n} x_i a_i$, where the vectors a_i are the columns

of A, $A = \begin{pmatrix} a_1 \dots a_n \end{pmatrix}$. Hence the matrix equation $Ax = b$ can be equivalently represented as $\sum_{j=1}^{n} x_i a_i = b$.

(b) **TRUE**
Justification: Consider a linear combination of vectors

$$\alpha_1 v_1 + \dots + \alpha_n v_n$$

If we denote $x_i = \alpha_i$ and $a_i = v_i$, then this combination is equivalent to $\sum_{j=1}^{n} x_i a_i$, which can be written as Ax, where $A = \begin{pmatrix} a_1 \dots a_n \end{pmatrix}$.

(c) **TRUE**
Justification: See Theorem 3 on page 42.

(d) **TRUE**
Justification: If $Ax = b$ is inconsistent, it means that there exists no x such that b can be represented as $\sum_{j=1}^{n} x_i a_i$, where a_i's are the columns of A, i.e. this means that b can not be represented as a linear combination of the columns of A. This is, however, equivalent to say that b is not in the span of these columns by definition of the span.

(e) **FALSE**
Justification: Consider, for example, the augmented matrix $[A\ x]$ such that:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

It has a pivot in each row, but it is consistent and has a solution.

(f) **TRUE**
Justification: By Theorem 4 on page 43 it follows that if Theorem 4 part (c) is **false** (i.e. columns of A do not span $\mathbb{R}^m$) then Theorem 4 part (a) is also **false**, which means that there exists a vector $b \in \mathbb{R}^m$ such that $Ax = b$ is not consistent.
Note, that the negation of the statement (a) of the theorem is **not** read as "every vector b", but rather states the existence of at least one.

2 page 48 #26

3 marks

Solution.
The matrix equation can be rewritten in the following way:

$$x_1 \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

and substituting for the u, v and w, we get

$$x_1 u + x_2 v = w$$

or

$$x_1 u + x_2 v - w = 0.$$

By comparing the coeffients, we obtain $x_1 = 3$ and $x_2 = -5$.

3 page 49 #40

3 marks

Solution.
Perform the row reduction, the columns span $\mathbb{R}^4$ if and only if each row has a pivot. The MATLAB program is as following:

```
A=[8 11 -6 -7 13
-7 -8 5 6 -9
11 7 -7 -9 -6
-3 4 1 8 7]
A(1,:)=A(1,:)/A(1,1)
A(2,:)=A(2,:)-A(2,1)*A(1,:)
A(3,:)=A(3,:)-A(3,1)*A(1,:)
A(4,:)=A(4,:)-A(4,1)*A(1,:)
A(2,:)=A(2,:)/A(2,2)
A(3,:)=A(3,:)-A(3,2)*A(2,:)
A(4,:)=A(4,:)-A(4,2)*A(2,:)
A(3,:)=A(3,:)/A(3,3)
A(4,:)=A(4,:)-A(4,3)*A(3,:)
A(4,:)=A(4,:)/A(4,4)
```

The result is:

```
A =

     8    11    -6    -7    13
    -7    -8     5     6    -9
    11     7    -7    -9    -6
    -3     4     1     8     7


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
   -7.0000   -8.0000    5.0000    6.0000   -9.0000
   11.0000    7.0000   -7.0000   -9.0000   -6.0000
   -3.0000    4.0000    1.0000    8.0000    7.0000
```

```
A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.6250   -0.2500   -0.1250    2.3750
   11.0000    7.0000   -7.0000   -9.0000   -6.0000
   -3.0000    4.0000    1.0000    8.0000    7.0000


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.6250   -0.2500   -0.1250    2.3750
         0   -8.1250    1.2500    0.6250  -23.8750
   -3.0000    4.0000    1.0000    8.0000    7.0000


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.6250   -0.2500   -0.1250    2.3750
         0   -8.1250    1.2500    0.6250  -23.8750
         0    8.1250   -1.2500    5.3750   11.8750


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.0000   -0.1538   -0.0769    1.4615
         0   -8.1250    1.2500    0.6250  -23.8750
         0    8.1250   -1.2500    5.3750   11.8750


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.0000   -0.1538   -0.0769    1.4615
         0         0         0         0  -12.0000
         0    8.1250   -1.2500    5.3750   11.8750


A =

    1.0000    1.3750   -0.7500   -0.8750    1.6250
         0    1.0000   -0.1538   -0.0769    1.4615
```

```
        0           0           0           0  -12.0000
        0           0           0      6.0000         0
Warning: Divide by zero.(Type "warning off MATLAB:
divideByZero" to suppress this warning.)
```

This result shows that $A(3,3)$ becomes zero, but we still could choose $A(3,5)$ and $A(4,4)$ as a pivot. Since each row has a pivot, we conclude that the columns of A span $\mathbb{R}^4$.

4 page 49 #42

3 marks

Solution.
In the solution of #40, we can see that the third column is not a pivot column, so we could delete the third column. Now each column has a pivot, so we can't delete more columns.

5 page 55 #6

3 marks

Solution.
We reduce the augmented matrix to the echelon form; the MATLAB program is as follows:

```
A=[1 3 -5 0
1 4 -8 0
-3 -7 9 0]
A(1,:)=A(1,:)/A(1,1)
A(2,:)=A(2,:)-A(2,1)*A(1,:)
A(3,:)=A(3,:)-A(3,1)*A(1,:)
A(2,:)=A(2,:)/A(2,2)
A(3,:)=A(3,:)-A(3,2)*A(2,:)
A(3,:)=A(3,:)/A(3,3)

A =

     1     3    -5     0
     1     4    -8     0
    -3    -7     9     0


A =

     1     3    -5     0
     1     4    -8     0
    -3    -7     9     0
```

```
A =

     1     3    -5     0
     0     1    -3     0
    -3    -7     9     0


A =

     1     3    -5     0
     0     1    -3     0
     0     2    -6     0


A =

     1     3    -5     0
     0     1    -3     0
     0     2    -6     0


A =

     1     3    -5     0
     0     1    -3     0
     0     0     0     0

Warning: Divide by zero.
(Type "warning off MATLAB:divideByZero" to suppress this warning.)
```

We can then convert it into the reduced echelon form:

$$A = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the solution has the form:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$$

6 page 55 #4

3 marks

Solution.
First exchange the rows 1 and 2, and the MATLAB program is:

```
A=[1 -2 6 0
-5 7 9 0]
A(1,:)=A(1,:)/A(1,1)
A(2,:)=A(2,:)-A(2,1)*A(1,:)
A(2,:)=A(2,:)/A(2,2)
A(1,:)=A(1,:)-A(1,2)*A(2,:)
```

The result is:

```
A =

     1    -2     6     0
    -5     7     9     0


A =

     1    -2     6     0
    -5     7     9     0


A =

     1    -2     6     0
     0    -3    39     0


A =

     1    -2     6     0
     0     1   -13     0


A =

     1     0   -20     0
     0     1   -13     0
```

Hence we obtain the solution:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20x_3 \\ 13x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 20 \\ 13 \\ 1 \end{pmatrix}$$

7 page 55 #14

3 marks

Solution.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 1 \\ -5 \\ 1 \end{pmatrix}$$

8 page 56 #34

3 marks

Solution.
By inspection, we can see that the second column is -1.5 times of the first column, so we can take the vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

as a non-trivial solution (you can verify that it is indeed a solution).