MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 10

1: Let P be the plane through $(-1,2,5)$ in the direction of the vectors $(1,-2,1)$ and $(0,1,-2)$, and let L be the line through $(3,5,-1)$ in the direction of $(1,2,-1)$.

(a) Find the standard equation of the plane P.

Solution: We have $(1,-2,1)\times(0,1,-2)=(3,2,1)$, so the equation is of the form $3x+2y+z=d$. Put in $(x,y,z)=(-1,2,5)$ to get $d=-3+4+5=6$, so the equation is $3x+2y+z=6$.

(b) Find the x, y and z intercepts of the plane P.

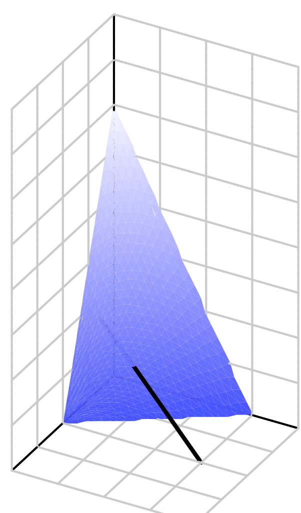
Solution: Put $y=z=0$ into the equation $3x+2y+z=6$ to get $3x=6$ so the x-intercept is $x=2$. Put in $x=z=0$ to get $2y=6$ so the y-intercept is $y=3$. Put in $x=y=0$ to get the z-intercept, $z=6$.

(c) Find the points of intersection of the line L with the xy-plane, with the xz-plane, with the yz plane, and with the plane P.

Solution: L is given by the vector equation $(x,y,z)=(3,5,-1)+t(1,2,-1)=(3+t,5+2t,-1-t)\quad(1)$. The intersection with the xy-plane occurs when $z=0$, that is when $-1-t=0$, so we put $t=-1$ back into equation (1) to get $(x,y,z)=(2,3,0)$. The intersection with the xz-plane occurs when $y=0$, that is when $5+2t=0$, so we put $t=-\frac{5}{2}$ into (1) to get $(x,y,z)=\left(\frac{1}{2},0,\frac{3}{2}\right)$. The intersection with the yz-plane occurs when $x=0$, that is when $3+t=0$, so we put $t=-3$ back into (1) to get $(x,y,z)=(0,-1,2)$. Finally, to find the point of intersection of L with P, we put $(x,y,z)=(3+t,5+2t,-1-t)$ into the equation $3x+2y+z=6$ to get $3(3+t)+2(5+2t)+(-1-t)=6$, so $6t=-12$, and so we put $t=-2$ back into (1) to get $(x,y,z)=(1,1,1)$.

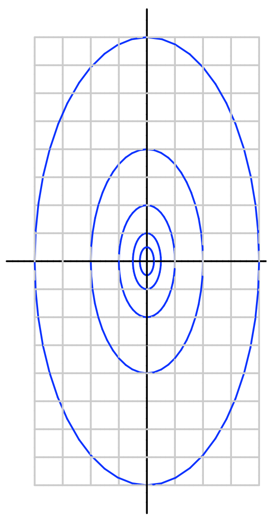
(d) Sketch the portion of the plane P which lies in the first octanct (given by $x\geq 0$, $y\geq 0$ and $z\geq 0$), together with the portion of the line L which lies in the first octant, showing their point of intersection.

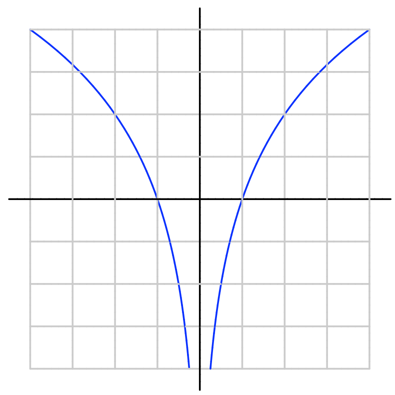
Solution: We can make the sketch using the information found in part (c).

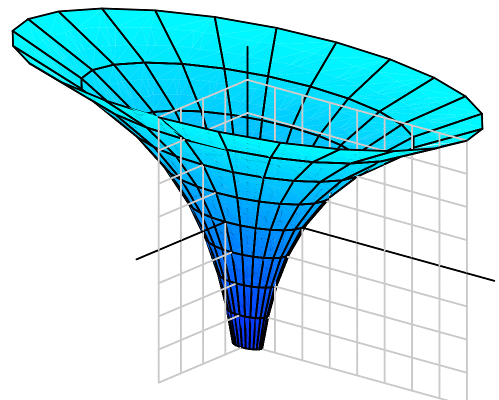


2: Sketch the level curves $z = -4, -2, 0, 2, 4$ and sketch the surface $z = \log_2(x^2 + \frac{1}{4}\,y^2)$.

Solution: The equation of the surface can be rewritten as $x^2 + \frac{1}{4}\,y^2 = 2^z$ or as $\dfrac{x^2}{2^z} + \dfrac{y^2}{4 \cdot 2^z} = 1$, so the level curve at z is the ellipse centered at 0 with major axis $2^{z/2}$ and minor axis $2 \cdot 2^{z/2}$. For $z = -4,\ -2,\ 0,\ 2$ and 4, the minor axis is $\frac{1}{4},\ \frac{1}{2}$, 1, 2 and 4, respectively. These level curves are sketched below at the left. We can find the curve of intersection of this surface with the xz-plane by putting $y = 0$ into the equation $z = \log_2(x^2 + \frac{1}{4}\,y^2)$ to get $z = \log_2(x^2) = 2\log_2|x|$, and this is sketched below in the center. Similarly, the curve of intersection of this surface with the yz-plane is given by $z = \log_2\left(\frac{1}{4}\,y^2\right) = 2\log_2\left|\frac{y}{2}\right|$. The surface is sketched below at the right.

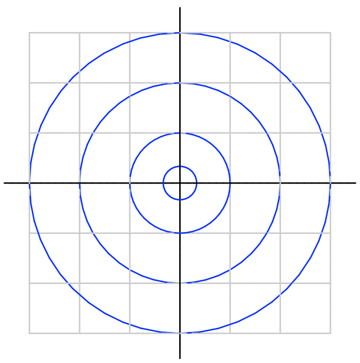


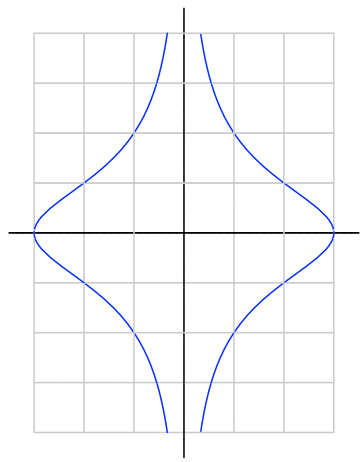


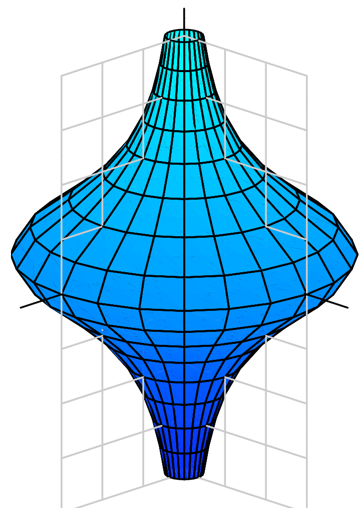


3: Sketch the level curves $z = -4, -2, -1, 0, 1, 2, 4$ and sketch the surface $x^2 + y^2 = \left(\dfrac{6}{2+z^2}\right)^2$.

Solution: The level curve at z is the circle centered at the origin of radius $\frac{6}{2+z^2}$. For $z = \pm 4$, ± 2, ± 1 and 0, the radius is $\frac{1}{3}$, 1, 2 and 3 respectively. These level curves are shown below at the left. To find the curve of intersection of the surface with the xz-plane, we put $y = 0$ into the equation of the surface to get $x^2 = \left(\frac{6}{2+z^2}\right)^2$, that is $x = \pm\frac{6}{2+z^2}$, which is shown below, in the center. Similarly, the curve of intersection of the surface with the yz-plane is $y = \pm\frac{6}{2+z^2}$. The surface is shown below, at the right.

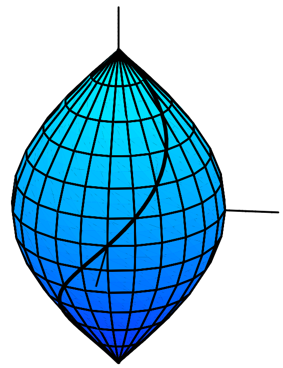






4: (a) Sketch the surface $x^2 + y^2 = \cos^2 z$ with $-\frac{\pi}{2} \le z \le \frac{\pi}{2}$.

Solution: The level curve at z is the circle centered at 0 with radius $\cos z$, and the curve of intersection of the surface with the yz-plane is $x = \pm \cos z$. The surface is shown below. Also, for your viewing pleasure, the curve followed by the ant is shown on the surface in black. Incidentally, and somewhat surprisingly, this curve is a helix which lies on the cylinder $x^2 + y^2 = x$.



(b) Show that the curve given by $(x, y, z) = (\cos^2 t, \sin t \cos t, t)$ with $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ lies on the above surface.

Solution: When $(x, y, z) = (\cos^2 t, \sin t \cos t, t)$, we have

$$x^2 + y^2 = \cos^4 t + \sin^2 t \cos^2 t = \cos^2 t\,(\cos^2 t + \sin^2 t) = \cos^2 t = \cos^2 z\,.$$

(c) An ant moves along the surface $x^2 + y^2 = \cos^2 z$ (where x, y and z are measured in inches) with its position at time t (in seconds) given by $(x, y, z) = (\cos^2 t, \sin t \cos t, t)$. Find the velocity and the acceleration of the ant when $t = \frac{\pi}{6}$.

Solution: The velocity is $v(t) = \big(x'(t), y'(t), z'(t)\big) = \big(-2\sin t \cos t, \cos^2 t - \sin^2 t, 1\big) = (-\sin 2t, \cos 2t, 1)$, and the acceleration is $a(t) = v'(t) = \big(x''(t), y''(t), z''(t)\big) = (-2\cos 2t, -2\sin 2t, 0)$. At time $t = \frac{\pi}{6}$ the velocity is $v\left(\frac{\pi}{6}\right) = \big(-\sin\frac{\pi}{3}, \cos\frac{\pi}{3}, 1\big) = \big(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\big)$ and the acceleration is $a(t) = \big(-2\cos\frac{\pi}{3}, -2\sin\frac{\pi}{3}, 0\big) = \big(-1, -\sqrt{3}, 0\big)$.

(d) Find the total distance travelled by the ant for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

Solution: The total distance travelled is the same as the arclength of the curve, and it is

$$L = \int_{-\pi/2}^{\pi/2} \big|v(t)\big|\,dt = \int_{-\pi/2}^{\pi/2} \sqrt{\sin^2 2t + \cos^2 2t + 1}\ dt = \int_{-\pi/2}^{\pi/2} \sqrt{2}\ dt = \Big[\sqrt{2}\,t\Big]_{\pi/2}^{\pi/2} = \sqrt{2}\,\pi\,.$$

5: Find a vector equation (or a parametric equation) for the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ with the sphere $x^2 + y^2 + z^2 = 4x$, then find a vector equation (or a parametric equation) for the tangent line to this curve at the point $(1, -1, \sqrt{2})$.

Solution: To find the top view of the curve of intersection, eliminate z by putting $z = \sqrt{x^2 + y^2}$ into the equation $x^2 + y^2 + z^2 = 4x$ to get $x^2 + y^2 + (x^2 + y^2) = 4x$, that is $x^2 + y^2 = 2x$. By completing the square, we can rewrite this as $(x - 1)^2 + y^2 = 1$, so from the top, the curve looks like the circle centered at $(x, y) = (1, 0)$ of radius 1. A parametric equation for this circle is $(x, y) = (1 + \cos t, \sin t)$ with $-\pi \le t \le \pi$. Put $x = 1 + \cos t$ and $y = \sin t$ into the equation $z = \sqrt{x^2 + y^2}$ to get $z = \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 + 2\cos t} = \sqrt{4\cos^2(t/2)} = 2\cos(t/2)$. Thus a vector equation for the curve of intersection is

$$(x, y, z) = r(t) = \big(1 + \cos t, \sin t, 2\cos(t/2)\big) \text{ with } -\pi \le t \le \pi\,.$$

The tangent vector is given by $v(t) = r'(t) = \big(-\sin t, \cos t, -\sin(t/2)\big)$. Note that when $t = -\frac{\pi}{2}$, we have $r\left(\frac{\pi}{2}\right) = (1, -1, \sqrt{2})$, and $v\left(-\frac{\pi}{2}\right) = \big(1, 0, \frac{\sqrt{2}}{2}\big)$, so a vector equation for the tangent line is

$$(x, y, z) = \big(1, -1, \sqrt{2}\big) + t\left(1, 0, \tfrac{\sqrt{2}}{2}\right).$$

The two surfaces are shown below, almost from the top view.