MATH 136 SOLUTIONS to Assignment 9
Fall/05
INCOMPLETE - Solutions will be finished ASAP.

Due: Monday Dec. 5/05.
(Grade is out of 46. In addition, there are 16 bonus marks.)

Contents

Part I
Questions

1 Page 254, Problem 20 *4 marks - page 2*

Let $w = k_1v_1 + \ldots + k_4v_4$ be an arbitrary vector in V. And, by the linear dependence assumption, we have $\underline{0} = c_1v_1 + \ldots + c_4v_4$, for some scalars c_i, not all zero. We can add these two equations to get the desired result, i.e.

$$w = w + \underline{0} = k_1v_1 + \ldots + k_4v_4 + c_1v_1 + \ldots + c_4v_4 = (k_1 + c_1)v_1 + \ldots + (k_4 + c_4)v_4$$

is a new expression for w since not all c_i are zero.

2 Page 254, Problem 24 *4 marks - page 2*

Let $y = (y_i) \in \mathbb{P}^n$. Define $u \in V$ with $u = \sum_{i=1}^n y_i\underline{b_i}$. Then, by definition, the coordinate map $[u]_B = y$.

3 Page 255, Problem 36 *4 marks - page 2*

(This can be done using MATLAB or by hand. But, in either case, show your work carefully.)
Using MATLAB with input:

```
!rm output
diary output
A=[-6 8 -9
4 -3 5
-9 7 -8
4 -3 3 ]
disp('reduced echelon form shows that the vectors are linearly indep.')
rref(A)
x=[4;7;-8;3]
disp('find the B-coordinate vector of x')
y=A\x
disp('show that x is a linear combination of columns of A')
x-A*y
diary off
```

The output from the MATLAB is:

```
A =

    -6     8    -9
     4    -3     5
```

```
    -9     7    -8
     4    -3     3

reduced echelon form shows that the vectors are linearly indep.

ans =

     1     0     0
     0     1     0
     0     0     1
     0     0     0


x =

     4
     7
    -8
     3

find the B-coordinate vector of x

y =

    3.0000
    5.0000
    2.0000

show that x is a linear combination of columns of A

ans =

  1.0e-014 *

   -0.7105
    0.3553
   -0.5329
    0.0888
```

4 Page 261, Problem 12,14 *6 marks - page 4*

12 The dimension of the subspace spanned by the four vectors is the column space of the following matrix

$$A = \begin{pmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{pmatrix}$$

By doing row reduction, it is clear that $dim(Col)(A)$ = number of pivot columns = 3. So the dimension of the given vectors is 3.

14 $dim(Col(A))$ = number of pivot columns = 3

$$Nul(A) = \{t \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -67 \\ 0 \\ -19 \\ -4 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 24 \\ 0 \\ -9 \\ -3 \\ 0 \\ 1 \end{pmatrix} |s, u, t \in R\}$$

Because these three vectors are linearly independent, $dim(Nul(A)) = 3$.

5 Page 261, Problem 22 *4 marks - page 4*

The four Laguerre polynomials are in $¶_3$ and the dimension of $¶_3$ is 4. Therefore, we only need to show linear independence to prove that they form a basis. Equivalently, we show that the coordinates vectors of the polynomials with respect to the standard basis are linearly independent, i.e. the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are linearly independent. This is clear since the matrix is upper triangular with nonzero diagonal elements.

6 Page 269, Problem 4 *4 marks - page 4*

There are three pivot columns, so the rank of A is 3. Since there are 6 columns, we conclude that dim Nul A is 3. The pivot columns can be chosen to be columns $1, 2, 4$ of B and so those columns of A form a basis for Col A. Also, the first 3 rows of A form a basis for Row A since the first 3 rows of B are linearly independent.

So a basis for Col(A):

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{pmatrix} \right\}$$

A basis for Row(A):

$\{(1 \quad 1 \quad -3 \quad 7 \quad 9 \quad -9), (1 \quad 2 \quad -4 \quad 10 \quad 13 \quad -12), (1 \quad -1 \quad -1 \quad 1 \quad 1 \quad -3)\}$

The basis for Nul(A) spans the general solution for $Ax = 0$, so we first transform B to reduced echelon form:

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis of Nul(A) is

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

7 Page 269, Problem 6 *6 marks - page 5*

Since A has three columns, and rank (A) =3, we conclude that the three columns of A must be linearly independent. Thus, we have:
dim (Col A) $= 3$
dim (Nul A) $= 3-$dim (Col A) $= 0$
Since A rank 3, we get A has exactly three pivots, so three rows have a pivot. Thus,
dim (Row A) $= 3$.
And (as always), rank $A^T =$ rank $A = 3$.

8 Page 270, Problem 28 *6 marks - page 5*

(a) dim Row A = number of pivots = number of pivot columns = dim Col A
dim Nul A = number of free variables = total number of columns - number of pivot columns = n - dim Col A
Thus dim Row A + dim Nul $A = n$; (This also follows from the rank theorem.)

(b) By the conclusion in (a), dim Nul A^T = number of columns in A^T - dim Row A^T = m- dim Row A^T
Because Row A^T = space spanned by rows of A^T = space spanned by columns of A = Col A.
Thus, dim Row A^T = dim Col A, and dim Col A + dim Nul A^T = m
(Again, apply the rank theorem.)

9 Page 276, Problem 12,14 *8 marks - page 6*

12 (a) In R^n, the change-of coordinate matrix from the $\mathcal{C}$ basis to the β basis is defined as:

$$P = (\,[b_1]_{\mathcal{C}} \quad [b_2]_{\mathcal{C}} \quad \dots \quad [b_n]_{\mathcal{C}}\,)$$

where $\beta = \{[b_1]_{\mathcal{C}}, [b_2]_{\mathcal{C}}, \dots, [b_n]_{\mathcal{C}}\}$ is the coordinate vectors of $b_1, b_2, \dots, b_n$ in basis $\mathcal{C}$, by the conclusion of page 254 # 25, if $b_1, b_2, \dots, b_n$ are linearly independent, then $[b_1]_{\mathcal{C}}, [b_2]_{\mathcal{C}}, \dots, [b_n]_{\mathcal{C}}$ are also linearly independent. Thus, the columns of $P\underline{\mathcal{C} \to \beta}$ are linearly independent.

(b) The row reduction to $[c_1, c_2, b_1, b_2]$ to $[IP]$ is equivalent to left-multiplier a matrix M, and $M[c_1, c_2] = I$, $M[b_1, b_2] = P$. By $M[c_1c_2] = I$, and c_1, c_2 is the basis for $\mathcal{C}$, we know $M^{-1} = [c_1, c_2]$, i.e. the change-of-coordinator matrix from standard basis to basis $\mathcal{C}$. So by $M[b_1, b_2] = P$, we get $[b_1, b_2] = M^{-1}P = [c_1, c_2]P$. Then,

$$\forall x \in R^n, x = [b_1, b_2][x]_\beta = [c_1, c_2][x]_{\mathcal{C}}$$

moreover

$$[b_1, b_2][x]_\beta = [c_1, c_2]P[x]_\beta$$

So

$$[c_1, c_2]P[x]_\beta = [c_1, c_2][x]_{\mathcal{C}}$$

by $[c_1, c_2]$ invertible, we get :

$$P[x]_\beta = [x]_{\mathcal{C}}$$

14 The change of coordinator matrix from β to standard basis $\mathcal{C}$ is the inverse of $(\,b_1 \quad b_2 \quad \dots \quad b_n\,)$, i.e.

$$\begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{pmatrix}$$

Part II
BONUS Questions

10 Page 382, Problem 16,18 *4 marks - page 7*

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$$u \cdot v = u^T v = \begin{pmatrix} 12 & 3 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = 24 - 9 - 15 = 0$$

So u and v are orthogonal.

18

$$y \cdot z = y^T z = \begin{pmatrix} -3 & 7 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -8 \\ 15 \\ -7 \end{pmatrix} = -3 - 56 + 60 + 0 = 1 \neq 0$$

So y and z are not orthogonal.

11 Page 383, Problem 30 *6 marks - page*

1. (a) Suppose $z \in W^{\perp} \subseteq R^n$. Then $\forall u \in W$, we get

$$z \cdot u = z^T u = 0$$

Therefore, $\forall c \in R, \forall u \in W$, we conclude

$$(cz) \cdot u = (cz)^T u = \sum_{i=1,2,\ldots n} c z_i u_i = c \sum_{i=1,2,\ldots n} z_i u_i = c(z^T u) = 0.$$

Thus, $cz \in W^{\perp}$ by the definition of $W^{\perp}$.

(b) Now suppose $z_1, z_2 \in W^{\perp}$, then $\forall u \in W$, we get

$$z_1 \cdot u = z_2 \cdot u = 0.$$

This implies $\forall u \in W$, we have

$$(z_1 + z_2) \cdot u = z_1 \cdot u + z_2 \cdot u = 0 + 0 = 0$$

So we conclude $z_1 + z_2 \in W^{\perp}$.

(c) Because

$$0 \cdot u = 0, \quad \forall u \in W,$$

, we have

$$0 \in W^{\perp},$$

by the definition of $W^{\perp}$.

Now, by the above we $0 \in W^{\perp}$ and also from a, b, $W^{\perp}$ is closed under scalar multiplication and also under vector addition. We conclude $W^{\perp}$ is a subspace of R^n.

12 Page 392, Problem 4,10,12 *6 marks - page*

4 Because:

$$\begin{pmatrix} 2 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & -5 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = 0$$

$$\begin{pmatrix} 4 & -2 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

We could claim the sets of vectors are orthogonal. (By the properties of the innerproduct, we only need to check either $u_i \cdot u_j$ or $u_j \cdot u_i$ and not both.)

10 Let

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$$

u_1, u_2, u_3. Then the $A^T A = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{pmatrix}$ a diagonal matrix, i.e. the columns are orthogonal and the matrix is rank n. Therefore, we can use the fact that the rank of a product AB is less or equal to the minimum of the ranks of A, B; and we conclude that A is nonsingular and so we can solve for the linear combination.

Alternatively: (by row reduction) A has a pivot in each column, so the columns are linearly independent; and there is a pivot in each row, so the three vectors span R^3. Thus, we know it is a basis of R^3.
Then, by checking:

$$u_1 \cdot u_2 = 0$$

$$u_1 \cdot u_3 = 0$$

$$u_2 \cdot u_3 = 0$$

We claim $\{u_1, u_2, u_3\}$ is an orthogonal set. And they also form a basis of R^3. We conclude they are an orthogonal basis for R^3. Thus $\forall x \in R^3$, x has unique coordinate for this orthogonal basis. Here $\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$ is the change-of-coordinate matrix from standard basis to the new orthogonal basis $\{u_1, u_2, u_3\}$. So we get:

$$[x]_\beta = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^{-1} x$$

$$= \begin{pmatrix} 3 & 2 & 1 \\ -3 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

So we could write

$$x = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$$

12.

The line through original and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ could be represented as $t\begin{pmatrix} -1 \\ 3 \end{pmatrix}, t \in R$. The orthogonal projection of vector y onto a vector u is

$$\frac{u \cdot y}{u \cdot u}u$$

So the orthogonal projection of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ onto any vector in this line except original $(t\begin{pmatrix} -1 \\ 3 \end{pmatrix}, t \in 0)$ is:

$$\frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot (t\begin{pmatrix} -1 \\ 3 \end{pmatrix})}{(t\begin{pmatrix} -1 \\ 3 \end{pmatrix}) \cdot (t\begin{pmatrix} -1 \\ 3 \end{pmatrix})}(t\begin{pmatrix} -1 \\ 3 \end{pmatrix}) =$$

$$\frac{-4}{10}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -1.2 \end{pmatrix}$$

which is independent with t. (Which tells us the orthogonal projection of a vector y onto another vector u doesn't vary if u is scaled.)