MATH 128 = Calculus 2 for the Sciences, Fall 2006
Assignment 3 SOLUTIONS

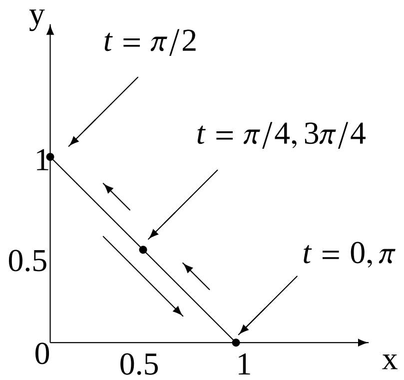
Due October 4 in Drop Box 9 before class

To receive full marks, correct answers must be fully justified.

1. (a) Sketch the curve $(x, y) = (\cos^2 t, \sin^2 t)$, for $0 \leq t \leq \pi$. Indicate with an arrow the direction in which the curve is traced as t increases. Also indicate points on the graph where $t = 0, \pi/4, \pi/2, 3\pi/4, \pi$.

   Solution: From the definitions of $x(t), y(t)$, we see that these terms will always be positive, hence the curve should only exist in quadrant I (for positive x and y). The restriction on t from 0 to π reveals that the first and last point is $(x, y) = (1, 0)$. The curve is a straight line segment traced from the point $(1, 0)$ toward the point $(0, 1)$ when $t = \pi/2$. Increasing t results in the reversal of direction back toward $(1, 0)$ where the curve ends. The arrows parallel to the straight line indicate the direction the curve is traced as t increases.

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
π	1	0



   (b) Determine the Cartesian equation, $y = f(x)$, of this curve. State it in explicit form.
   Solution: Since $x + y = \cos^2 t + \sin^2 t = 1$, then $\boxed{y = 1 - x}$.

Aside: also acceptable: solve $x(t)$ for t to obtain $t = \cos^{-1}(\pm\sqrt{x})$, checking the restriction on $\pm\sqrt{x}$ (where we apply the principal values to $\cos^{-1}$), which is $-1 \leq \pm\sqrt{x} \leq 1$ which can be simplified to $-1 \leq \sqrt{x} \leq 1$ and then $0 \leq x \leq 1$. (This restriction limits $\cos^{-1}(\pm\sqrt{x})$ to values between 0 and $\pi/2$ inclusive.) Then substitute this t value into $y(t)$ to obtain $y(x) = \sin^2(\cos^{-1}(\pm\sqrt{x}))$, which is limited to $y \in [0, 1]$ (since the argument of sin has been limited to $[0, \pi/2]$). We are not yet done: we must check if there is any algebraic simplification possible of our expression for $y(x)$.

Consider $\sin^2(\cos^{-1}(A))$. Let $B = \cos^{-1} A$. Then $\cos B = A$. From a right-angled triangle with the values A and B denoted accordingly, we see that the hypotenuse length is 1 and the opposite side to angle B is $\sqrt{1 - A^2}$, according to the Pythagorean theorem. Therefore, we deduce that $\sin B = \sqrt{1 - A^2}$, hence $\sin^2 B = 1 - A^2$. For our problem, this means that $y = \sin^2(\cos^{-1}(\pm\sqrt{x}))$ must be simplified to $y = 1 - (\pm\sqrt{x})^2 = 1 - x$.

2. The parametric equations $x = a\cos^3\theta, y = a\sin^3\theta$ trace out what is called a *hypocycloid of four cusps*, or an *astroid*. Figure 1 is a graph of an astroid for $a = 1$ and $0 \le \theta \le 2\pi$.

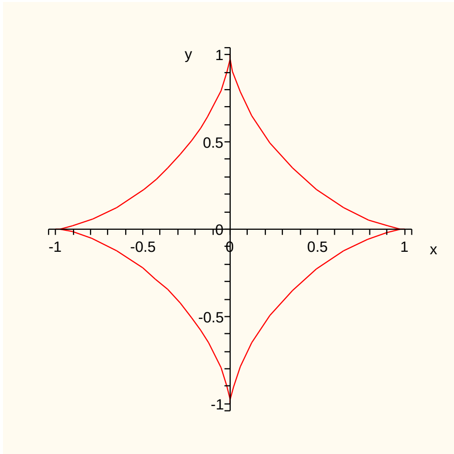


Figure 1: Astroid $(x, y) = (\cos^3(\theta), \sin^3(\theta))$

(a) Calculate the arc length of the curve in Figure 1. Your integral must be in terms of the parameter θ. [Hints: (1) You can use symmetry. (2) Use the identity $\cos^2\theta + \sin^2\theta = 1$ in simplifying the integrand.]

Solution: The curve is symmetric, so we need only compute the length of one fourth of the entire curve and then multiply that amount by four to obtain the total arc length L. We choose, for simplicity, the segment from $\theta = 0$ to $\theta = \pi/2$. We need the derivatives of $x(\theta)$ and $y(\theta)$:

$$x = \cos^3\theta \Rightarrow \frac{dx}{d\theta} = 3\cos^2\theta(-\sin\theta), \quad y = \sin^3\theta \Rightarrow \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

$$\begin{aligned}
\Rightarrow L &= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 4\int_0^{\pi/2} \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\, d\theta \\
&= 4\int_0^{\pi/2} \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\, d\theta \\
&= 4\int_0^{\pi/2} \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, d\theta \\
&= 4\int_0^{\pi/2} \sqrt{9\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta = 4\cdot 3\int_0^{\pi/2} \sqrt{\cos^2\theta\sin^2\theta}\, d\theta \\
&= 12\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta
\end{aligned}$$

Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$. Then $\theta = 0 \Rightarrow u = 0, \theta = \pi/2 \Rightarrow u = 1$:

$$\Rightarrow L = 12\int_0^1 u\, du = 12\left[\frac{u^2}{2}\right]_0^1 = 12\cdot\frac{1}{2} = \boxed{6}.$$

Aside: Regarding the simplification of $\sqrt{\cos^2\theta\sin^2\theta}$:
Note that $\sqrt{\cos^2\theta\sin^2\theta} = \cos\theta\cdot\sin\theta$ only because $\cos\theta$ and $\sin\theta$ are both positive for $0 \le \theta \le \pi/2$. However, if either of the factors were negative, then special care has to be taken when taking its square root. For example, $\sqrt{(-2)^2\cdot(3)^2} = \sqrt{4\cdot 9} = \sqrt{36}$ which, if seen on its own, one would likely simplify to +6. But we know here from the start that we began with the square of one negative factor and of one positive factor, which means the correct answer would be -6: $\sqrt{(-2)^2\cdot(3)^2} = \sqrt{(-2\cdot 3)^2} = \sqrt{(-6)^2} = -6$.
For comparison, try using limits of integration as $\pi/2$ to π, which correspond to the segment of the curve in quadrant II. In this case $\sqrt{\cos^2\theta\sin^2\theta} = -\cos\theta\cdot\sin\theta$. Your result for L will be off by a negative sign if you did not take proper care in this particular simplification to allow that $\cos\theta$ is negative in this region.

(b) Set up, but do not evaluate, an integral in terms of θ representing the area of the enclosed region defined by $(x, y) = (\cos^3\theta, \sin^3\theta)$. (Your final answer can use symmetry.)

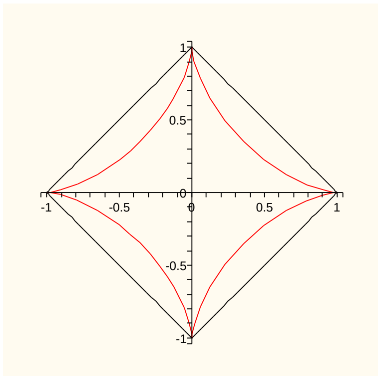
Solution: Let $x(\theta) = f(\theta) = \cos^3\theta$ and $y(\theta) = g(\theta) = \sin^3\theta$. Then the area A of the enclosed region is given by:

$$\begin{aligned} A &= \int_0^{2\pi} g(\theta) f'(\theta) d\theta \\ &= 4\int_0^{\pi/2} \sin^3\theta(3\cos^2\theta)(-\sin\theta)\,d\theta \\ &= -12\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta \end{aligned}$$

where $f'(\theta) = -3\cos^2\theta\sin\theta$ and we have used symmetry of the problem by restating the integral (in the second line) to represent one-fourth of the total area, hence the factor of 4.

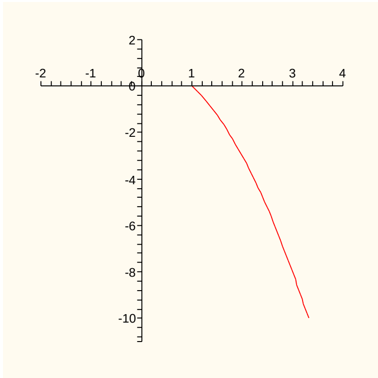
(c) Without solving your integral from part b, what is a reasonable upper limit (*i.e.*, a maximum) for the area of the enclosed region? (Hint: consider a simple geometric shape that this astroid can fit inside whose area is easy to calculate.)

Solution: The astroid easily fits into a square with vertices as shown in the figure to the right. The square's side length is $\sqrt{2}$, so its total area is $(\sqrt{2})^2 = 2$. Thus, $\boxed{2}$ is a reasonable maximum value for the area of the region enclosed by the astroid.



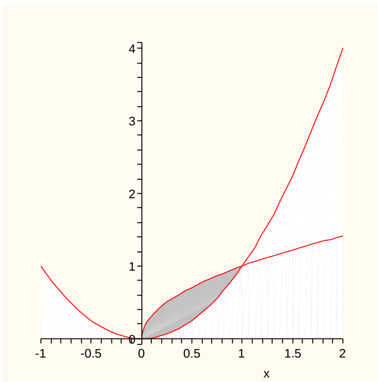
3. Sketch the curve $(x, y) = (\sqrt{s+1}, -s)$ for $0 \leq s \leq 10$. Indicate with an arrow the direction in which the curve is traced as s increases. (Pay attention to the restriction on s and how it affects the curve.)

Solution: Since $x^2 = s + 1 \Rightarrow s = x^2 - 1 \Rightarrow y = -s = 1 - x^2 = (1 - x)(1 + x)$. By their definitions and restriction on s, we know that $-10 \leq y \leq 0$ and that $1 \leq x \leq \sqrt{11}$. We can deduce which direction the curve is traced either from a table of values or from examining $x(s)$ and $y(s)$. Since $\sqrt{(\cdot)}$ is an increasing function, so $x(s)$ must increase as s increases. Likewise, y obviously decreases as s increases, since $y = -s$. So the first point is $(x, y) = (1, 0)$ and the last point is $(x, y) = (\sqrt{11}, -10)$.



4. Let $\mathcal{R}$ be the region bounded by $y = x^2$ and $y = \sqrt{x}$. Find the volume of the solid obtained by rotating $\mathcal{R}$ about the $x-$axis using:
   (a) cylindrical shells.
   (b) washers

Solution: The region $\mathcal{R}$ is the gray shaded region shown in the figure, where the top of the gray region is the curve $y = \sqrt{x}$ and the bottom is given by $y = x^2$. The intersection points of the two curves are (0,0) and (1,1). Let V denote the volume of the solid of revolution.



(a) cylindrical shells:
Cylindrical shells will be lined up parallel to the $x-$axis, in the $y-$direction. Hence the integration is in terms of y. (Imagine a horizontal strip of thickness Δy or dy in the region $\mathcal{R}$ that touches both curves. This strip will rotate 360^o around the $x-$axis to *carve out* a cylindrical shell or *tin can* in three dimensions.)

- The **radius** of any shell is the $y-$distance from the axis of rotation, the $x-$axis, to the midpoint of a shell of thickness Δy or dy. This distance is just an arbitrary distance in y, hence we consider this radius to be simply y.

- The **height** $h(y)$ of any shell is given as the *top function* - *bottom function*, expressed in terms of y. The *top* function is the one farthest right when looking at the graph in the normal way—with positive x values on the right, negative x on the left, and positive y at the top, negative y on the bottom. But, since we're rotating the region around the x−axis, it may help you to rotate the graph of $\mathcal{R}$ 90^o counterclockwise so that the axis of rotation is vertical (such that positive x values are on the top, and negative y values are to the right). Then the *top function* is now literally the top curve, $y = x^2$. And the *bottom function* is clearly $y = \sqrt{x}$. We need $h(y)$, so express these functions in terms of y: $y = x^2 \Rightarrow x = \pm\sqrt{y}$ but we know $x \geq 0$, so choose $x = +\sqrt{y}$. And $y = \sqrt{x} \Rightarrow x = y^2$. Hence, $\boxed{h(y) = \sqrt{y} - y^2}$.
- The **limits of integration** must be in terms of y. Coincidentally here, the limits are the same in x and y, 0 to 1. (This is not always necessarily the case.)

$$\begin{aligned} V &= 2\pi \int_0^1 yh(y)dy = 2\pi \int_0^1 y(\sqrt{y} - y^2)dy = 2\pi \int_0^1 y^{3/2} - y^3 dy \\ &= 2\pi \left[\frac{2}{5}y^{5/2} - \frac{y^4}{4}\right]_0^1 dy = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) = 2\pi\left(\frac{8}{20} - \frac{5}{20}\right) = \frac{3\pi}{10} \end{aligned}$$

(b) washers:

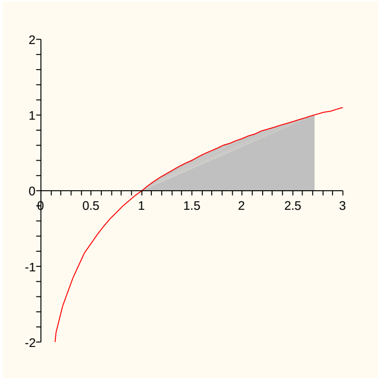
Washers will be lined up in the x−direction. Hence the integration is in terms of x. (Imagine a vertical strip of thickness Δx or dx in the region $\mathcal{R}$ that touches both curves. This strip will rotate 360^o around the x−axis to *carve out* a washer in three dimensions.)

- The **inner radius** r_i of the washer is the distance from the axis of rotation, the x−axis, to the inner edge of $\mathcal{R}$. (Again, you may find it helpful to rotate the graph so the x−axis is vertical.) The inner curve is given by $y = x^2$, hence $r_i(x) = x^2$.
- The **outer radius** r_o of the washer is the distance from the axis of rotation, the x−axis, to the outer edge of $\mathcal{R}$. The outer curve is given by $y = \sqrt{x}$, hence $r_o(x) = \sqrt{x}$.
- The **limits of integration** must be in terms of x: 0 to 1.

$$\begin{aligned} V &= \pi \int_0^1 (r_o{}^2 - r_i{}^2)\, dx = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\, dx = \pi \int_0^1 x - x^4\, dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \pi\left(\frac{5}{10} - \frac{2}{10}\right) = \frac{3\pi}{10} \end{aligned}$$

5. Let $\mathcal{R}$ be the region bounded by $y = \ln(x)$ and $y = 0$ for $1 \le x \le e$.
   (a) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the $x-$axis.
   (b) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the $y-$axis.

Solution: The region $\mathcal{R}$ is the gray shaded region shown in the figure. Let V denote the volume of the solid of revolution.



(a) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the $x-$axis.

Since $\mathcal{R}$ rests along the axis of rotation, the $x-$axis, we can use disks stacked along the $x-$axis to calculate V. Hence integration is in terms of x. (Imagine a vertical strip of thickness Δx or dx in $\mathcal{R}$ that touches the $x-$axis and $\ln(x)$. This strip will rotate 360^o around the $x-$axis to *carve out* a circular disk in three dimensions.)

- The **radius** of a disk is the distance from the axis of rotation, the $x-$axis, to the outer edge of $\mathcal{R}$, given by $y = \ln(x)$. Hence $r(x) = \ln(x)$.
- The **limits of integration** in x are from 1 to e.

$$V = \pi \int_1^e r(x)^2\,dx = \pi \int_1^e (\ln(x))^2\,dx$$

Use integration by parts: $u = (\ln(x))^2 \Rightarrow du = \dfrac{2\ln(x)}{x}dx$ and $dv = dx \Rightarrow v = x$:

$$\Rightarrow V = \pi\left\{\left[x(\ln(x))^2\right]_1^e - 2\int_1^e \ln(x)\,dx\right\}$$

Use integration by parts again: $u = \ln(x) \Rightarrow du = \dfrac{dx}{x}$ and $dv = dx \Rightarrow v = x$:

$$\begin{aligned}
\Rightarrow V &= \pi\left\{(e(\ln(e))^2 - 1\cdot(\ln(1))^2) - 2\left[x\ln(x) - x\right]_1^e\right\} \qquad (\ln(e) = 1, \ln(1) = 0)\\
&= \pi[e - 2(e\ln(e) - e - (1\ln(1) - 1))]\\
&= \pi(e - 2e + 2e - 2)\\
&= \pi(e - 2)
\end{aligned}$$

(b) Find the volume of the solid obtained by rotating $\mathcal{R}$ about the $y-$axis.

Since there is a gap between $\mathcal{R}$ and the axis of rotation, the $y-$axis, we must choose between washers and cylindrical shells. Either method works nicely here, since for shells there is one function that acts as the *top* and one as the *bottom* for all $x \in [1, e]$; and for washers there is one function that acts as the inner radius and one as the outer for all $y \in [0, 1]$. (For shells, imagine a vertical strip of thickness Δx or dx that touches $\ln(x)$ and the $x-$axis. For washers, imagine a horizontal strip of thickness Δy or dy that touches $\ln(x)$ and the vertical line $x = e$.)

Solutions for shells and washers are presented.

Solution by cylindrical shells:

Shells will be lined up in the $x-$direction, thus integration is in terms of x. The height of the shell is $h(x) = \ln(x) - 0 = \ln(x)$. The radius is the distance from the axis of rotation, the $y-$axis, to an arbitrary shell. This distance is simply an x distance, hence $r(x) = x$.

$$V = 2\pi \int_1^e x h(x)\, dx = 2\pi \int_1^e x \ln(x)\, dx$$

Use integration by parts: $u = \ln(x) \Rightarrow du = \dfrac{dx}{x}$ and $dv = x dx \Rightarrow v = \dfrac{x^2}{2}$:

$$\begin{aligned} \Rightarrow V &= 2\pi \left\{ \left[\frac{\ln(x)}{2} x^2 \right]_1^e - \int_1^e \frac{x^2}{2} \frac{1}{x}\, dx \right\} = 2\pi \left\{ \frac{e^2}{2} \ln(e) - \frac{\ln(1)}{2} - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e \right\} \\ &= 2\pi \left(\frac{e^2}{2} - \frac{1}{2} \left(\frac{e^2}{2} - \frac{1}{2} \right) \right) = 2\pi \left(\frac{e^2}{4} + \frac{1}{4} \right) = \frac{\pi}{2}(e^2 + 1) \end{aligned}$$

Solution by washers:

Washers will be lined up in the $y-$direction, thus integration is in terms of y. Since $y = \ln(x)$, we rewrite the curve as $x = e^y$. The inner radius of the washer is the distance from the axis of rotation, the $y-$axis, to the inner edge of $\mathcal{R}$: $r_i(y) = e^y$. The outer radius of the washer is the distance from the $y-$axis to the outer edge of $\mathcal{R}$: $r_o(y) = e$.

$$\begin{aligned} V &= \pi \int_0^1 ({r_o}^2 - {r_i}^2)\, dy = \pi \int_0^1 (e^2 - (e^y)^2)\, dy \\ &= \pi \int_0^1 e^2 - e^{2y}\, dy = \pi \left[e^2 y - \frac{e^{2y}}{2} \right]_0^1 \\ &= \pi \left(e^2 - \frac{e^2}{2} - \left(e^2 \cdot 0 - \frac{e^0}{2} \right) \right) = \pi \left(\frac{e^2}{2} + \frac{1}{2} \right) = \frac{\pi}{2}(e^2 + 1) \end{aligned}$$