MATH 128 Calculus 2 for the Sciences, Solutions to Assignment 3

1: (a) Find a solution of the form $y = ax^2 + bx + c$ to the DE $y''y' + x^2 = y$.

Solution: Let $y = ax^2 + bx + c$. Then $y' = 2ax + b$ and $y'' = 2a$ and so we have $y''y' + x^2 = y \iff 2a(2ax + b) + x^2 = ax^2 + bx + c \iff x^2 + 4a^2\,x + 2ab = ax^2 + bx + c$. Equating coefficients gives $1 = a$, $4a^2 = b$ and $2ab = c$, and so we must have $a = 1$, $b = 4$ and $c = 8$. Thus the only such solution is $y = x^2 + 4x + 8$.

(b) Find constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\,e^{r_1 x} + b\,e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

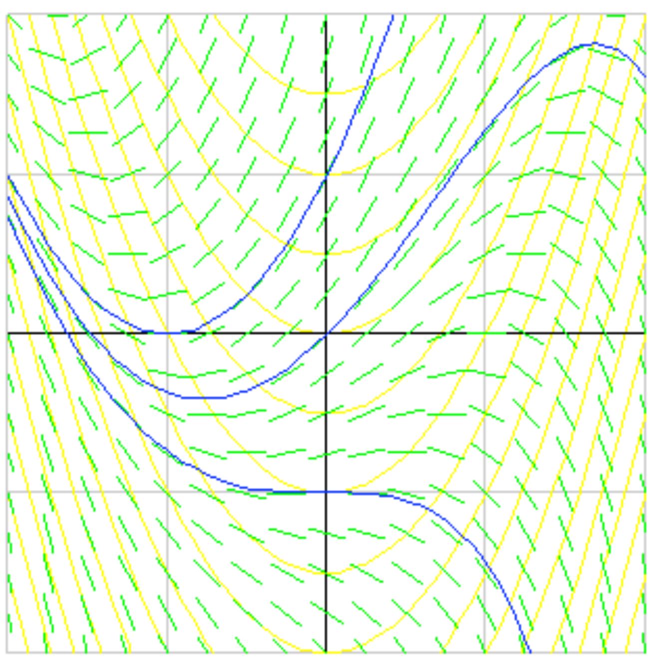
Solution: Let $y = e^{rx}$. Then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$ and so $y''+3y'+2y = 0 \iff r^2\,e^{rx}+3r\,e^{rx}+2\,e^{rx} = 0 \iff (r^2 + 3r + 2)e^{rx} = 0 \iff (r + 1)(r + 2)\,e^{rx} = 0 \iff r = -1$ or $r = -2$. Thus we can take $r_1 = -1$ and $r_2 = -2$.

Now, let $y = a\,e^{r_1 x} + b\,e^{r_2 x} = a\,e^{-x} + b\,e^{-2x}$. Then $y' = -a\,e^{-x} - 2b\,e^{-2x}$ and $y'' = a\,e^{-x} + 4b\,e^{-2x}$ and so we have $y'' + 3y' + 2y = a\,e^{-x} + 4b\,e^{-2x} - 3a\,e^{-x} - 6b\,e^{-2x} + 2a\,e^{-x} + 2b\,e^{-2x} = 0$. This shows that $y = a\,e^{-x} + b\,e^{-2x}$ is a solution to the DE. Also, note that $y(0) = a + b$ and $y'(0) = -a - 2b$, and so to get $y(0) = 1$ and $y'(0) = 0$ we need $a + b = 1$ and $-a - 2b = 0$. Solve these two equations to get $a = 2$ and $b = -1$. Thus the required solution is $y = 2\,e^{-x} - e^{-2x}$.

2: (a) Sketch the direction field of the DE $y' = y - x^2 + 1$.

(b) On the same grid, sketch the solution curve with $y(0) = 1$, the solution curve with $y(0) = 0$ and the solution curve with $y(0) = -1$.

Solution: The isoclines, which are the parabolas $m = y - x^2 + 1$ where m is the slope, are shown below in yellow, the direction field is shown in green, and the three solution curves are shown in blue.



3: Solve the following differential equations.

(a) $x\,y' + 2y = \sqrt{1+x^2}$

Solution: This DE is linear since we can write it in the form $y' + \dfrac{2}{x}y = \dfrac{\sqrt{1+x^2}}{x}$. An integrating factor is $\lambda = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$ and the solution is $y = \frac{1}{x^2}\displaystyle\int x\sqrt{1+x^2}\,dx = \frac{1}{x^2}\big(\frac{1}{3}(1+x^2)^{3/2} + c\big)$.

(b) $y' = y^2 + 3y + 2$

Solution: This DE is separable. Note that $y^2+3y+2 = (y+1)(y+2)$. When $y \neq -1$ or -2 we can write the DE as $\dfrac{y'}{(y+1)(y+2)} = 1$ and integrate both sides. We find that y is a solution $\iff \displaystyle\int \frac{dy}{(y+1)(y+2)} = \int dx$ $\iff \displaystyle\int \frac{1}{y+1} - \frac{1}{y+2}\,dy = x + c_1 \iff \ln|y+1| - \ln|y+2| + c_2 = x + c_1 \iff \ln\left|\frac{y+1}{y+2}\right| = x + c \iff$ $\left|\dfrac{y+1}{y+2}\right| = e^{x+c} = e^c e^x \iff \dfrac{y+1}{y+2} = \pm e^c e^x = a\,e^x \iff y+1 = ya\,e^x + 2a\,e^x \iff y\,(1 - a\,e^x) = 2a\,e^x - 1$ $\iff y = \dfrac{2a\,e^x - 1}{1 - a\,e^x}$, where c_1 and c_2 are arbitrary constants, and we have written $c = c_1 - c_2$ and then $a = \pm e^c$. Since the constant functions $y = -1$ and $y = -2$ are both solutions to the DE, the general solution is given by $y = -2$ or $y = \dfrac{2a\,e^x - 1}{1 - a\,e^x}$, where a is arbitrary.

4: Solve the following initial value problems.

(a) $(1+x^2)y' = x\,y$ with $y(0) = 2$.

Solution: This DE is both separable and linear, and we choose to solve it as a separable DE. When $y \neq 0$ we can write the DE as $\dfrac{y'}{y} = \dfrac{x}{1+x^2}$ then integrate both sides. Since we are interested in the solution with $y(0) = 2$ we shall consider the case that $y > 0$. Then y is a solution $\iff \ln y = \frac{1}{2}\ln(1+x^2) + c \iff$ $y = e^{\frac{1}{2}\ln(1+x^2)+c} = a\sqrt{1+x^2}$, where $a = e^c$. The condition $y(0) = 2$ then gives $2 = a$ and so $y = 2\sqrt{1+x^2}$.

(b) $x^2y' - y = 1$ with $y(1) = 1$.

Solution: This DE is both separable and linear and we choose to solve it as a linear DE. We write it as $y' - \dfrac{1}{x^2}\,y = \dfrac{1}{x^2}$. An integrating factor is $\lambda = e^{\int -\frac{1}{x^2}\,dx} = e^{1/x}$ and the solution to the DE is given by $y = e^{-1/x}\displaystyle\int \frac{1}{x^2}\,e^{1/x}\,dx = e^{-1/x}\big(-e^{1/x} + c\big) = -1 + c\,e^{-1/x}$. Put in $y(1) = 1$ to get $1 = -1 + c/e$ so $c = 2e$ and the solution to the IVP is $y = -1 + 2\,e^{1-1/x}$.

5: Solve the IVP $y'' + 2y' = 4x$ with $y(0) = 0$ and $y'(0) = 0$. Hint: let $u = y'$ so $u' = y''$.

Solution: Let $u = y'$ so that $u' = y''$. Then the DE becomes $u' + 2u = 4x$. This is linear; an integrating factor is $\lambda = e^{\int 2\,dx} = e^{2x}$ and the solution is $u = e^{-2x}\displaystyle\int 4x\,e^{2x}\,dx$. We integrate by parts using $u = 4x$, $du = 4\,dx$, $v = \frac{1}{2}\,e^{2x}$ and $dv = e^{2x}\,dx$ to get $u = e^{-2x}\left(2x\,e^{2x} - \displaystyle\int 2\,e^{2x}\,dx\right) = e^{-2x}\big(2x\,e^{2x} - e^{2x} + c\big) = 2x - 1 + c\,e^{-2x}$. To get $u(0) = 0$ we need $-1 + c = 0$ so $c = 1$, and so we have $u = 2x - 1 + e^{-2x}$, that is $y' = 2x - 1 + e^{-2x}$. Integrate to get $y = x^2 - x - \frac{1}{2}\,e^{-2x} + d$. To get $y(0) = 0$ we need $-\frac{1}{2} + d = 0$ so $d = \frac{1}{2}$, and so the solution is $y = x^2 - x - \frac{1}{2}\,e^{-2x} + \frac{1}{2}$.