Introduction to Beam Physics and Accelerator Technology
Dr. Giulio Stancari
University of Ferrara
April–May 2022

Homework

The homework for this course consists of definitions, exercises and problems. It is an integral part of the course. The goals are to review the concepts discussed in class, to make quantitative estimates of phenomena and design parameters, and to explore topics not covered during the lectures. Of course, the homework is also used to assess and evaluate whether learning objectives have been achieved.

Definitions should range in length from one sentence to one paragraph. Focus on the core concept and its main properties. Avoid formulas or use them sparingly.

Exercises and problems should include descriptions of the logical procedure, variable definitions, assumptions, approximations, special cases, etc., not only formulas and numbers.

Some parts of the homework are optional. You will not lose points if you do not complete them, but they will give you extra points if done correctly.

Solutions can be prepared individually or as a group. Discussions and interactions are encouraged.

Each student should turn in a scan of his or her own *handwritten* copy to receive credit. (If possible, e-mail me a link rather than a large attachment.)

The deadline to turn in your homework is **Tuesday, June 21.**

1. Define the following terms: magnetic rigidity; cyclotron frequency.

2. Consider a quadrupole magnet with half gap $R = 25$ mm and length $l = 20$ cm.

   (a) Find the total current NI that is necessary to generate a gradient $B' = 10$ T/m.

   (b) What is the focal length of this magnet for 10-GeV/c protons?

3. (optional) (a) For a relativistic particle of mass m, write the de Broglie wavelength λ as a function of kinetic energy T.

   (b) Plot the wavelength λ as a function of T for electrons and protons, for T varying between 10 eV and 10 TeV. Use logarithmic scales for both abscissa and ordinate.

   (c) On the vertical scale of wavelengths, identify the typical sizes of physical systems, such as cells, molecules, atoms, and nuclei. On the horizontal scale of kinetic energies, indicate the typical energies of a few accelerators of your choice, such as electron microscopes, betatrons, AdA, SLAC linac, LEP, cyclotrons, Bevatron, AGS, Tevatron, LHC, etc.

4. (a) Calculate the center-of-momentum energy of a projectile A (mass m_A, total energy E_A) and a stationary target B of mass m_B.

   (b) Find the proton kinetic energy that is necessary to generate proton-antiproton pairs on a stationary hydrogen target, $p + p \to p + p + \bar{p} + p$.

   (c, optional) What beam energies were used in the discovery of the antiproton at the Bevatron [1]? Comment.

5. Copy and complete the following table:

Particle	Electron e^-	Proton p	Gold ion $^{197}\mathrm{Au}^{79+}$
Charge, q [e]	-1	$+1$	$+79$
Rest energy, mc^2	0.511 MeV	0.938 GeV	197 u
Kinetic energy, T	1 MeV	1 GeV	1 GeV/nucleon
Momentum, pc [GeV]			
Velocity parameter, $\beta = v/c$			
Magnetic rigidity, $(B\rho)$ [T m]			

6. Define the following terms: synchrotron; synchrotron radiation.

7. In a fixed-target experiment, every 60 s ($= 1/f$), a Gaussian bunch with $N_p = 10^{11}$ protons completely crosses a liquid hydrogen target (mass density $\delta = 0.07$ g/cm^3, thickness $L = 1$ m). The dimensions of the proton bunch are small compared to the size of the target.

   (a) Show that the integrated luminosity over one crossing is $N_p\, n_t\, L$, where n_t is the number density of protons in the target. Calculate the numerical value of the average luminosity $\langle \mathscr{L} \rangle = f\, N_p\, n_t\, L$.

   (b) At these energies, the total pp cross section is $\sigma_{\mathrm{tot}} = 40$ mb. Calculate the total number of events expected in each crossing.

   (c) The goal of this experiment is to study a rare process with cross section $\sigma_i \simeq 1$ pb. For this purpose, it is necessary to collect a total of $N_e = 10^5$ of these rare events. What is the minimum duration of the experiment?

8. Consider a realistic model of the operation cycle of a collider. Starting at $t = 0$, data is taken for a time t_s, which can be chosen by the experimenters. During this time, due to beam lifetime, emittance growth and other factors, luminosity decays exponentially with time constant t_L. At time t_s, the beam is dumped, the magnets are ramped down, more beam is injected, the beam energy is ramped up again, and the beams are put back into collision. These operations require a time t_o, known as the turn-around time, which is fixed. During the turn-around time, the luminosity is zero and no events are recorded. At time $t_s + t_o$, data taking restarts and the cycle is repeated.

   (a) Find an expression for the average integrated luminosity over one cycle.

   (b) Determine whether it is possible to optimize the data-taking time t_s in order to maximize the average integrated luminosity.

   (c) Test this optimization algorithm with a numerical example: $t_L = 9$ h, $t_o = 2$ h. What is the optimal data-taking time t_s in this case?

9. Define the following terms: phase stability; phase-slip factor.

10. The Fermilab Booster is a synchrotron used to accelerate protons from a kinetic energy of 400 MeV up to 8 GeV. Its circumference is 468 m. The accelerating radiofrequency (rf) cavities operate at the 84th harmonic of the revolution frequency. Calculate by how much the revolution period, revolution frequency, and rf frequency vary during the acceleration cycle.

11. In the Tevatron, protons and antiprotons were accelerated from a kinetic energy of 150 GeV to 980 GeV in 60 s. The circumference was $L = 6.28$ km and the transition energy was $\gamma_t = 18.7$. The radiofrequency cavities operated at 53.1 MHz. The maximum energy they could provide was $qV = 1.4$ MeV per revolution.

    (a) Calculate the energy gain per revolution, assuming it was constant during acceleration.

    (b) Calculate the corresponding synchronous phase ϕ_s.

    (b) For both the injection energy and the maximum energy, calculate the synchrotron tune ν_s and the synchrotron frequency f_s. For small oscillation amplitudes, how many revolutions were necessary to complete a synchrotron oscillation?

12. Define the following terms: alternating gradient; emittance; dynamic aperture.

13. In a collider like the LHC, the beta function at the interaction region is $\beta = 30$ cm and the rms beam emittance is $\varepsilon = 0.3$ nm. What is the standard deviation of the transverse beam size at the interaction region?

14. A generic linear beam transport system can be represented by simple equivalent systems consisting of a drift, a thin lens and another drift (OFO' system) or by a thin lens, a drift, and another thin lens (FOF' system). To show this, consider a known system represented by the transport matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

with $\det M = 1$.

(a) Find the matrix $M_{OFO'}$ describing a system consisting of a drift of length L_1, a thin lens with focal length f and another drift of length L_2.

(b) Assume $m_{21} \neq 0$. Determine the values of L_1, f and L_2 for which $M_{OFO'} = M$.

(c) In the case $m_{21} = 0$, the system M is called *telescopic*. What does it mean?

(d) Find the matrix $M_{FOF'}$ describing a system consisting of a thin lens of focal length f_1, a drift of length L and another thin lens of focal length f_2.

(e) Determine f_1, L and f_2 such that $M_{FOF'} = M$.

(f) When both $M_{OFO'}$ and $M_{FOF'}$ can be determined from M, do the simple equivalent systems have the same length?

15. This problem is about self fields and space-charge forces in particle beams. Consider a beam of protons, each of mass $m = 1.67 \times 10^{-27}$ kg and charge $q = 1.60 \times 10^{-19}$ C, all moving along the $\hat{\mathbf{z}}$ direction with the same velocity $v = 2.5 \times 10^8$ m/s.

If the number of particles is very large, the charge density of the beam can be approximated by a continuous function ρ. In this case, the charge density is cylindrically symmetrical, it is Gaussian with standard deviation σ in the plane perpendicular to the direction of motion, and it is independent of z and time. In cylindrical coordinates,

$$\rho(r, \theta, z, t) = \rho(r) = \frac{q\lambda}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right),$$

where $\lambda = 150\ \mathrm{m}^{-1}$ is a constant representing the number of particles per unit length.

(a) Write an expression for the current density $\mathbf{j}$ and the total beam current I generated by the beam. Calculate the numerical value of the beam current.

(b) Derive an expression for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ generated by the beam, specifying their direction.

(c) Calculate the Lorentz force $\mathbf{F}$ acting on each particle by adding the electic part $q\mathbf{E}$ and the magnetic part $q\mathbf{v} \times \mathbf{B}$. Sketch a plot of the magnitude of the force as a function of r.

(d) In beam physics, this force is called the 'space-charge' force. Show that it is proportional to r for $r \ll \sigma$ and to $1/r$ for $r \gg \sigma$. Show that this force decreases with beam energy by exposing its dependence on the relativistic factor $\gamma \equiv (1 - v^2/c^2)^{-1/2}$.

16. (optional) Using a programming language of your choice with graphic capabilities (R, python, ROOT, ...), write a program to calculate and plot the phase space coordinates in the (x, p) plane of a dynamical system described by Chirikov's standard map:

$$\begin{cases} p_{n+1} &= p_n + K \sin x_n \\ x_{n+1} &= x_n + p_n + K \sin x_n = x_n + p_{n+1} \end{cases}$$

for several different initial conditions and a few different values of the constant parameter K. You may choose the maximum number of iterations $n_{\max}$, $0 \le n \le n_{\max}$. One may consider both x and p modulo 2π and restrict the analysis to the square $0 \le x/2\pi \le 1$ and $0 \le p/2\pi \le 1$. Notice how the dynamics becomes chaotic when $K \simeq 0.97$. This standard map serves as a model for many physical systems. Attach a printout of your program and a few representative plots.

References

[1] Owen Chamberlain et al. "Observation of Antiprotons." In: *Phys. Rev.* 100.3 (Nov. 1955), pp. 947–950. DOI: 10.1103/PhysRev.100.947. URL: https://link.aps.org/doi/10.1103/PhysRev.100.947.