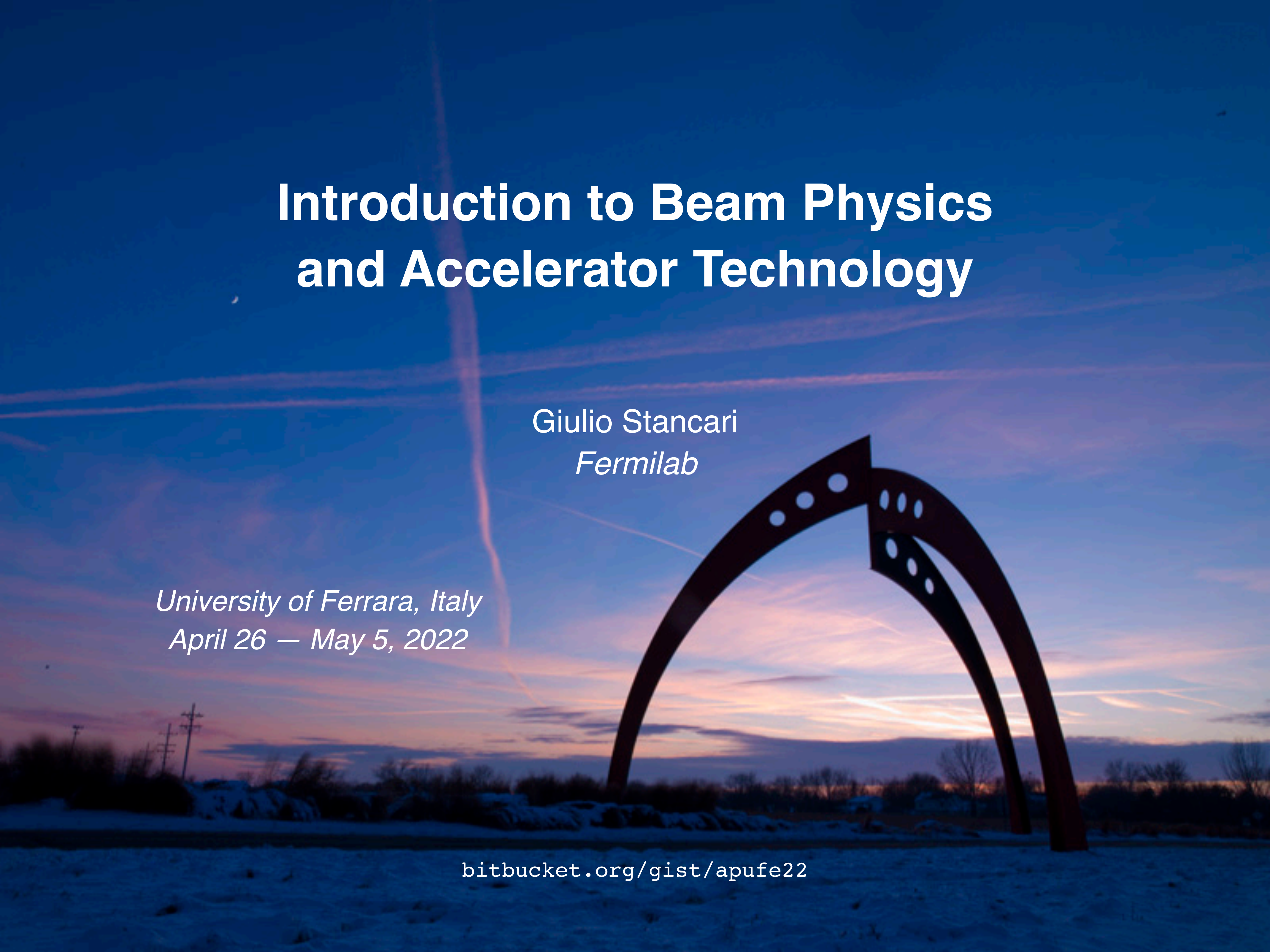


Introduction to Beam Physics and Accelerator Technology

Giulio Stancari

Fermilab

University of Ferrara, Italy

April 26 — May 5, 2022

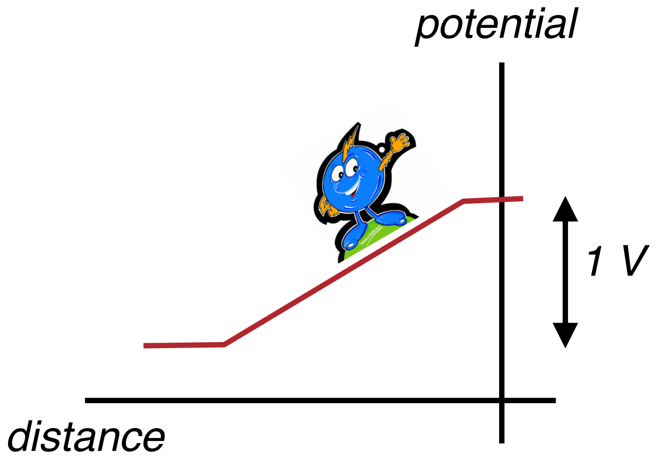
bitbucket.org/gist/apufe22

Review of concepts in
mechanics, relativity, electromagnetism

Units of energy

The **electronvolt** (eV) is a useful unit of energy

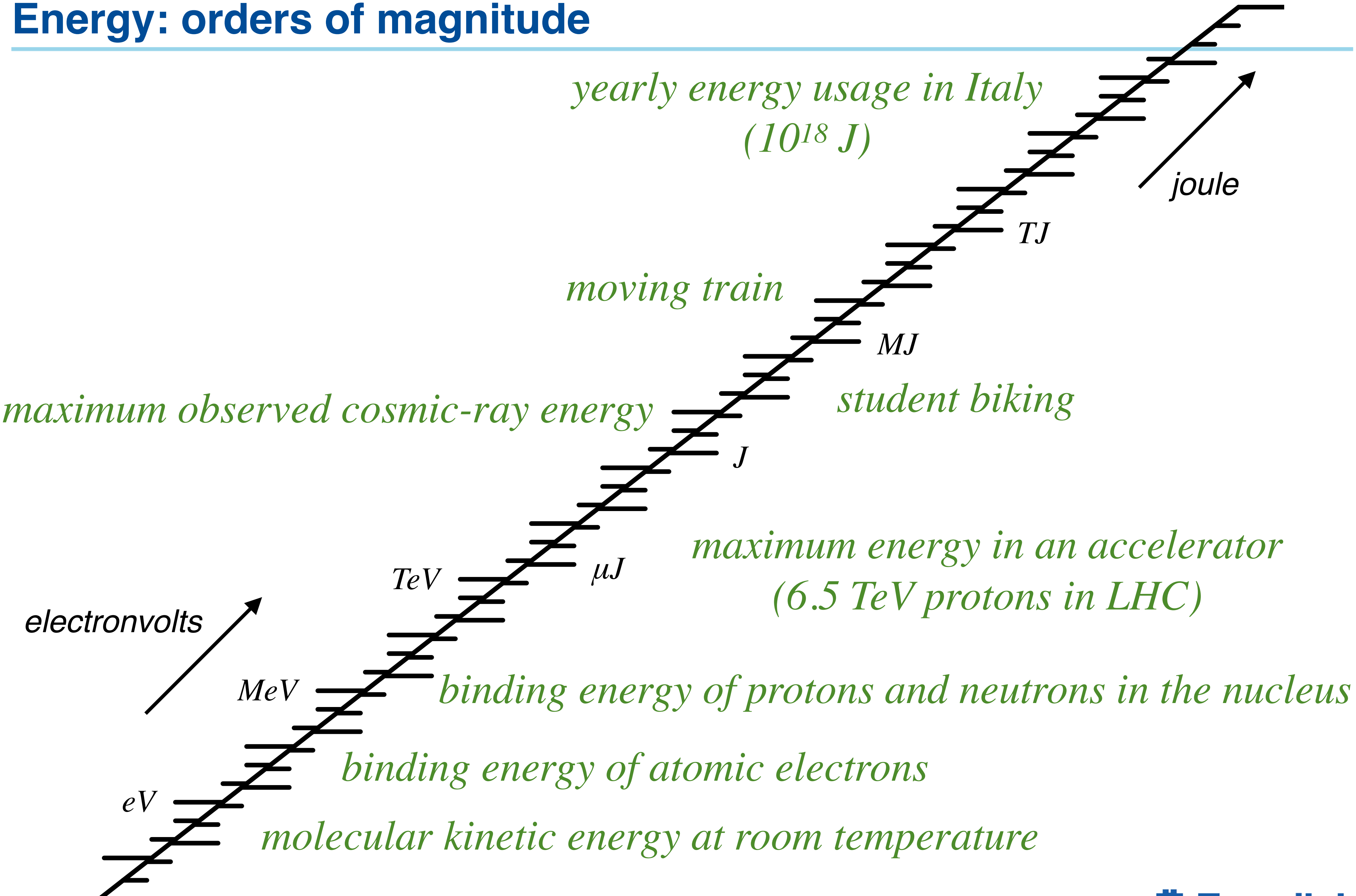
It corresponds to the kinetic energy acquired by an electron traversing a potential difference of 1 V



$$1\ \text{eV} \equiv q \cdot \Delta V = (1.602 \times 10^{-19}\ \text{C}) \cdot (1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$$

Multiples are the keV (10^3 eV), MeV (10^6 eV), GeV (10^9 eV), TeV (10^{12} eV), etc.

Energy: orders of magnitude



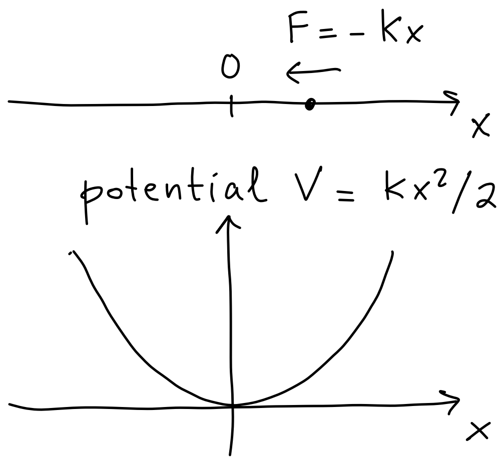
Formulations of dynamics: a summary

$\dot{x} \equiv \frac{dx}{dt} \qquad \partial_x f \equiv \frac{\partial f}{\partial x}$

	Newtonian	Lagrangian	Hamiltonian
coordinates	spatial $\mathbf{x}$	generalized $(q, \dot{q})$	canonical pairs $(q, p = \partial_{\dot{q}}\mathscr{L})$
characteristic functions	forces	Lagrangian $\mathscr{L}(q, \dot{q}) = T - V$	Hamiltonian $H(q, p) = p\dot{q} - \mathscr{L}$
equations of motion	$\mathbf{F} = \dot{\mathbf{p}}$	$\frac{d}{dt}\left(\partial_{\dot{q}}\mathscr{L}\right) - \partial_q\mathscr{L} = 0$	$\begin{cases} \dot{q} = \partial_p H \\ \dot{p} = -\partial_q H \end{cases}$
main features	links forces and variations in momenta (Newton's second law)	generalized coordinates are not necessarily orthogonal takes *k* constraints into account	canonical transformations may simplify problems concept of phase space
	N 2nd order differential eq.	*(N—k)* 2nd order differential eq.	2*(N-k)* 1st order differential eq.

Example: 1-dimensional harmonic oscillator

linear **force**, i.e. proportional to displacement



momentum $p = m\dot{x}$

Newton's equation of motion

$$F = \dot{p}$$
$$-kx = m\ddot{x}$$
$$\ddot{x} + (k/m)x = 0$$

solution

$$x(t) = A \cdot \cos\left(\omega t + \phi\right)$$

constant amplitude and phase
from initial conditions
$x(0),\ \dot{x}(0)$

constant **oscillation frequency** and **period**

$$\omega = \sqrt{\frac{k}{m}} \qquad \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

independent of oscillation amplitude

Example: 1-dimensional harmonic oscillator

Lagrangian approach

simplest choice of **generalized coordinates**

$$q = x \quad \dot{q} = \dot{x}$$

kinetic energy $T = m\dot{q}^2/2$

potential energy $V = kq^2/2$

Lagrangian → same **equation of motion**

$$\mathscr{L}(q, \dot{q}) = T - V$$
$$= m\dot{q}^2/2 - kq^2/2$$

$$\frac{d}{dt}\left(\partial_{\dot{q}}\mathscr{L}\right) - \partial_q \mathscr{L} = 0$$

$$m\ddot{q} + kq = 0$$

Example: 1-dimensional harmonic oscillator

Hamiltonian approach

same choice of **generalized coordinate** $q = x$

canonical momentum $p \equiv \partial_{\dot{q}} \mathscr{L} = m\dot{q}$

Hamiltonian

$$\begin{aligned} H &= p\dot{q} - \mathscr{L} \\ &= p^2/m - p^2/(2m) + kq^2/2 \\ &= p^2/(2m) + kq^2/2 \\ &= T + V \end{aligned}$$

→ **equations of motion**

$$\begin{cases} \dot{q} = \partial_p H = p/m \\ \dot{p} = -\partial_q H = -kq \end{cases}$$

The Hamiltonian is **constant**

$$\begin{aligned} \dot{H} &= \partial_q H \cdot \dot{q} + \partial_p H \cdot \dot{p} + \partial_t H \\ &= kq\dot{q} + p\dot{p}/m \\ &= kqp/m - kqp/m = 0 \end{aligned}$$

→ **Energy is conserved**

→ In **phase space** *(q, p)*, the system evolves on an **ellipse**

$$\frac{q^2}{2H/k} + \frac{p^2}{2mH} = 1$$

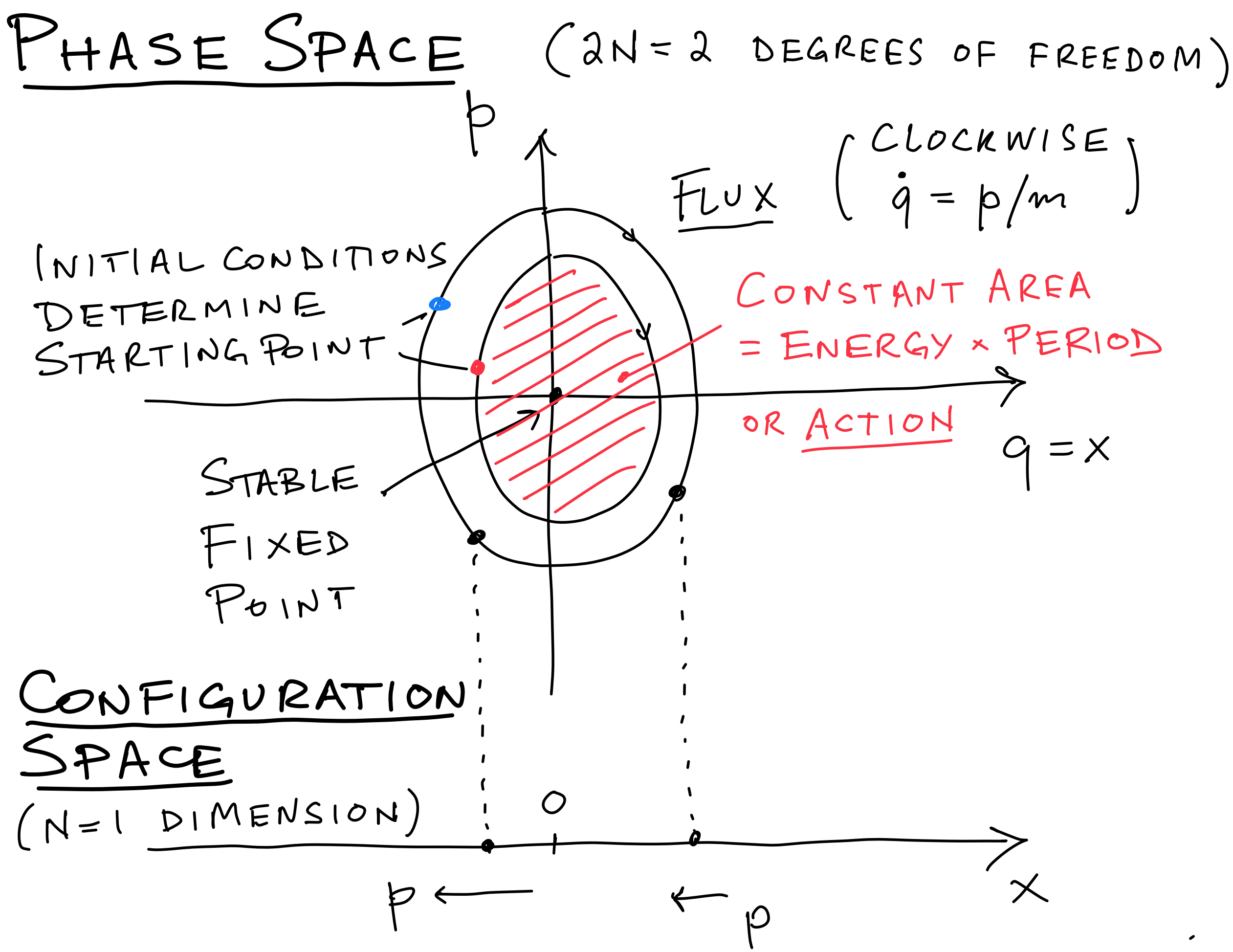
with **constant area** $\pi\sqrt{\frac{2H}{k} 2mH} = H \cdot \tau$

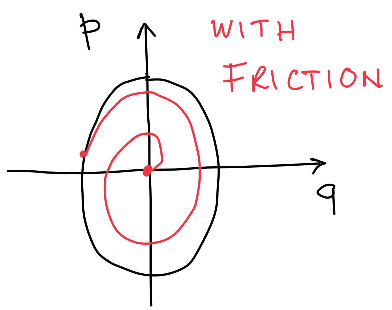
Ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Area $= \pi a b$

Hamiltonian approach and the system's phase-space topology





The **graphical representation** of the system summarizes its **dynamics**

Areas and volumes in phase space are related to the concept of **emittance** in beam physics

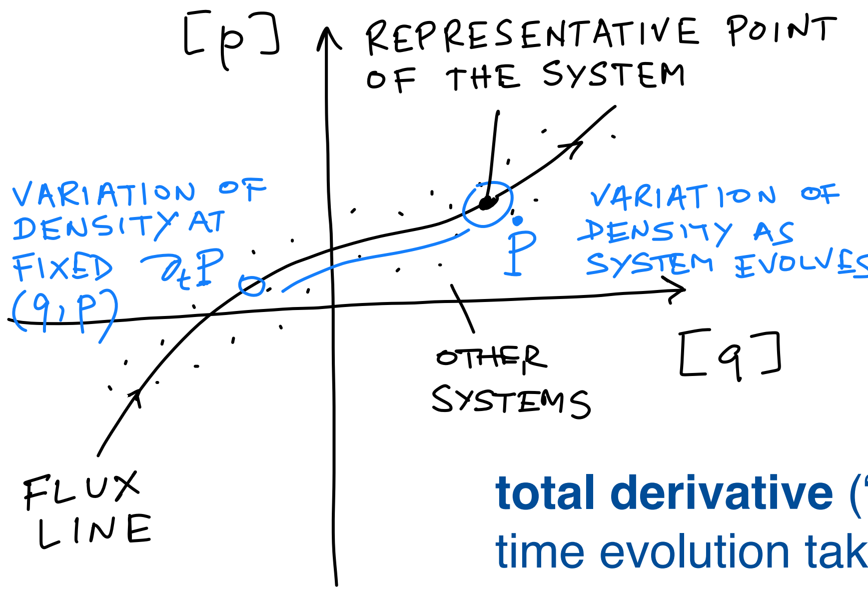
Statistical description of dynamical systems

System with many particles or many equivalent systems (“ensemble”)

density of systems
$$P(q, p)$$

number of systems
$$P(q, p)\, dq\, dp$$

in phase space volume
$$dq = dq_1\, dq_2\, \ldots dq_{3N}$$
$$dp = dp_1\, dp_2\, \ldots dp_{3N}$$



partial derivative $\partial_t P$
change at **constant *q* and *p***

total derivative (“hydrodynamic derivative”) $\dot{P}$
time evolution taking into account the variation of *q* and *p*, i.e. the **dynamics of the system**

Examples

- **Beam** of $N = 10^9$ non-interacting protons
  - N representative points (“particles”) in 6-dimensional phase space or
  - 1 representative point (“whole system”) in $6N$-dimensional phase space
- Statistical ensemble of L **ideal gases** with $N = 10^{23}$ particles each
  - L representative points in $6N$-dimensional phase space (huge!)

Evolution of phase-space density and Liouville's theorem

if dynamics is Hamiltonian

$$\begin{cases} \dot{q} = \partial_p H \\ \dot{p} = -\,\partial_q H \end{cases}$$

The density of states along a flux line is constant

continuity equation $\quad \partial_t P + \partial_q (P \dot{q}) + \partial_p (P \dot{p}) = 0$

$$\partial_t P + (\partial_q P)\dot{q} + P \partial_q \dot{q} + (\partial_p P)\dot{p} + P \partial_p \dot{p} = 0$$

$$\partial_t P + (\partial_q P)\dot{q} + (\partial_p P)\dot{p} + P\left(\partial_q \partial_p H - \partial_p \partial_q H\right) = 0$$

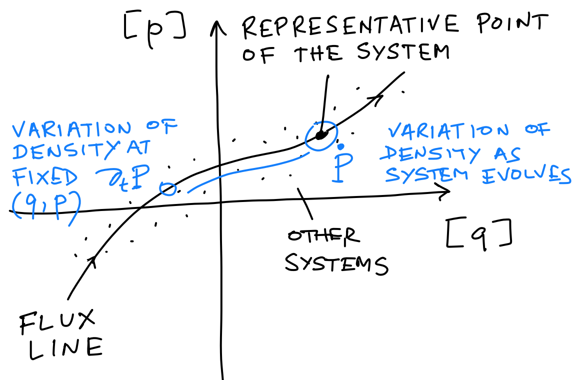
$$\dot{P} = 0$$

if number of systems is constant (continuity equation)

$$\partial_t P + \nabla \cdot (P \cdot \mathbf{v}) = 0$$

$\nabla \cdot \mathbf{f} = \partial_q f_q + \partial_p f_p \qquad \mathbf{v} = (\dot{q}, \dot{p})$

divergence and velocity in phase space



The density of states can change due to nonconservative forces or energy exchanges with the environment

In beam physics, phase-space density variations can be due to

- "heating": scattering on residual gas, intrabeam scattering, internal targets, …
- "cooling": synchrotron radiation damping, electron cooling, stochastic cooling, …

Questions?

Special relativity

Principle of relativity: physical laws must have the same form in all inertial reference frames

COVARIANCE or **INVARIANCE IN FORM** of **physical laws**

Example: Newton's second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \rightarrow \mathbf{F}' = \frac{d\mathbf{p}'}{dt'}$$

Example: Gauss's law

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \rightarrow \nabla' \cdot \mathbf{E}' = \rho'/\epsilon_0'$$

A **physical quantity** is **INVARIANT** when it has the same numerical value in all reference frames

Example: electric charge

$$Q = Q'$$

Example: speed of light in vacuum

$$c = c'$$

Relativistic kinematics and dynamics

velocity parameter $\beta = \frac{v}{c}$

$0 \leq \beta \leq 1$

Lorentz factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = (1 - \beta^2)^{-1/2}$ $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$

$1 \leq \gamma$

4-vectors $x^\mu \equiv (ct, \mathbf{x}) \equiv (ct, x, y, z)$ $x_\mu \equiv (ct, -\mathbf{x}) \equiv (ct, -x, -y, -z)$ $\mu = 0,1,2,3$

4-vector components in different inertial systems change according to **Lorentz transformations**

The **contraction** of 4-vectors is **invariant**

example $x_\mu x^\mu \equiv \sum_{\mu=0}^{3} x_\mu x^\mu \equiv (ct)^2 - x^2 - y^2 - z^2$ space-time interval

Relativistic kinematics and dynamics

4-momentum of a particle $p^\mu \equiv (E/c, \mathbf{p}) \equiv (E/c, p_x, p_y, p_z)$ $\mathbf{p} \equiv \gamma m \mathbf{v}$

For convenience, we often redefine **masses and momenta in units of energy**

$$mc^2 \to m$$
$$pc \to p$$

Total energy

$$E = \gamma m$$

Kinetic energy

$$T = (\gamma - 1)m$$

Velocity parameter equals **momentum / energy ratio** $\beta = \dfrac{p}{E}$

Contractions of 4-momenta

$p_\mu p^\mu = E^2 - p^2 = m^2$ **rest energy of a particle** (invariant)

total 4-momentum of a system

$P^\mu \equiv p_1^\mu + p_2^\mu + \ldots$

$P_\mu P^\mu$ **center-of-momentum energy** (invariant)

Relativistic dynamics

With the **relativistic definition of momentum** $\mathbf{p} \equiv \gamma m \mathbf{v}$

Newton's second law can still be written

$$\mathbf{F} = \dot{\mathbf{p}}$$

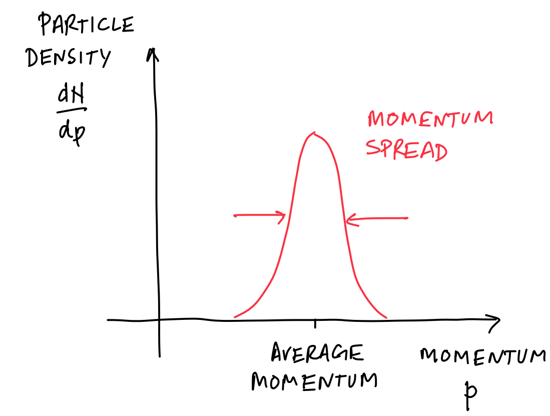
For constant magnitude of the velocity $\dot{\gamma} = 0$
(uniform circular motion, for instance)

$$\mathbf{F} = \dot{\mathbf{p}} = \gamma m \mathbf{a}$$

Same as classical equation, replacing $m \to \gamma m$

Relative momentum, energy and velocity spread of a beam

To understand **dynamics in accelerators** and to **design experiments**, it is often essential to know the **range of kinematic beam parameters**



RELATIVE MOMENTUM SPREAD
(largest)

$$\frac{dp}{p} = \frac{1}{\beta^2}\frac{dE}{E}$$

$$\frac{dE}{E} = \beta^2 \frac{dp}{p}$$

RELATIVE ENERGY SPREAD
(intermediate)

$$\frac{dp}{p} = \gamma^2 \frac{dv}{v}$$

$$\frac{dv}{v} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{dE}{E} = (\beta\gamma)^2 \frac{dv}{v}$$

RELATIVE VELOCITY SPREAD
(smallest)

$$\frac{dv}{v} = \frac{1}{(\beta\gamma)^2}\frac{dE}{E}$$

Fermilab

Example of energy, momentum and velocity spreads

The relative beam energy spread of 6.5-TeV protons in the LHC is about 1 x 10^{-4}

What is the relative momentum spread?

What is the relative velocity spread?

Example of numerical calculation (SageMathCell)

Questions?

Electromagnetism is described by Maxwell's equations

Electric **charge** **CONSERVED** continuity eq. $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$

INVARIANT value independent of reference frame

Charges and **currents** are **sources** of fields ⟷ **Fields** act on **charges** and **currents**

	differential form	integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$	$\int_S \mathbf{E} \cdot \hat{\mathbf{n}} = Q_S / \epsilon_0$
Law of induction	$\nabla \times \mathbf{E} = - \partial_t \mathbf{B}$	$\oint \mathbf{E} \cdot \hat{\mathbf{t}} \, ds = - \dot{\Phi}_B$
No magnetic charges	$\nabla \cdot \mathbf{B} = 0$	$\int_S \mathbf{B} \cdot \hat{\mathbf{n}} = 0$
Ampère's law	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \cdot \partial_t \mathbf{E} \right)$	$\oint \mathbf{B} \cdot \hat{\mathbf{t}} \, ds = \mu_0 \cdot i_{\text{tot}}$

$$\mu_0 \epsilon_0 = 1/c^2$$

Charged particle in electromagnetic field: Lorentz force

A major part of accelerator physics is devoted to the study of the motion of charged particles in electromagnetic fields

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

Electric fields are used to accelerate and to deflect

Static **magnetic fields** can only deflect: force is perpendicular to velocity

Effective at high energy: force proportional to velocity

High-energy accelerators use **magnets** for **confinement** and **focusing**

Example: electric vs. magnetic forces

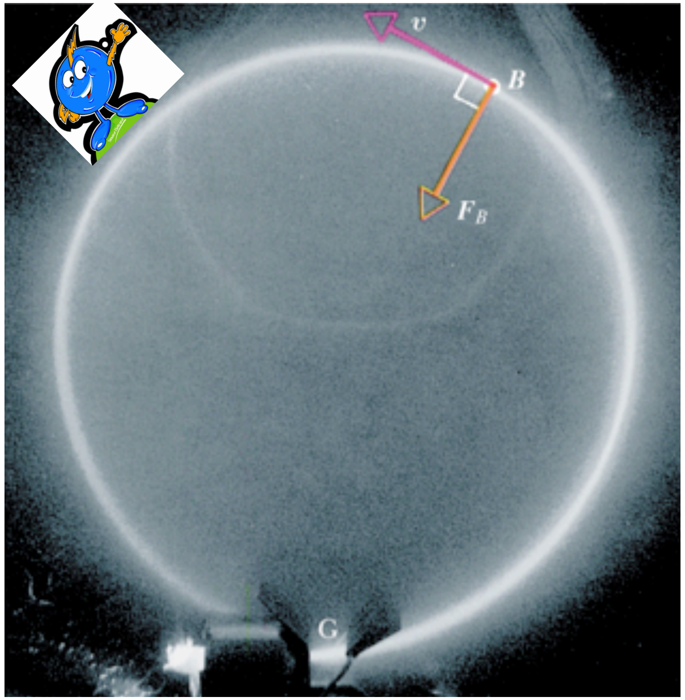
At what velocities do magnetic fields become more effective?

(a) Choose typical values for strong electric and magnetic fields that can be obtained in the laboratory

(b) For these values of the fields, at which velocity does the magnetic force become more intense than the electric force?

Charged particle in constant magnetic field

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$



1-keV electron source in 1-mT magnetic field.
Radius of curvature: 11 cm.

Momentum parallel to the magnetic field is conserved

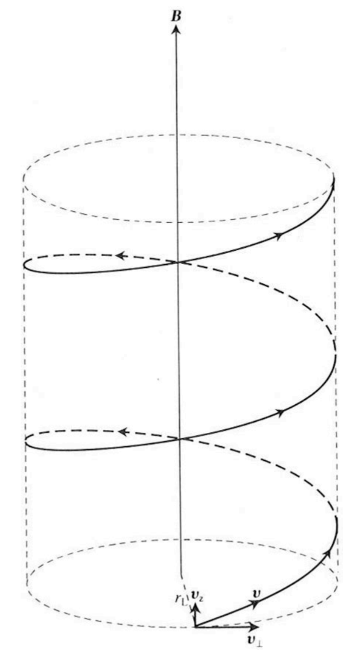
$$\dot{p}_{\parallel} = 0$$

In the **plane perpendicular to the field**, momentum $p \equiv p_{\perp}$ changes direction but not magnitude: **uniform circular motion**

Helical path with **radius** $\rho = \dfrac{p}{qB}$

and **angular frequency** $\omega = \dfrac{qB}{\gamma m}$

"cyclotron frequency"



$$\frac{v^2}{\rho} = \frac{qvB}{\gamma m}$$

$$\frac{p}{q} = B\rho$$

magnetic rigidity ("B rho") *of the particle*
= momentum / charge

Numerical estimates of magnetic rigidity

A convenient **conversion factor** (adimensional constant equal to 1)

$$\frac{B\rho q}{p} = 1$$

$$1 = \frac{(1\ \mathrm{T})\cdot(1\ \mathrm{m})\cdot(1\ \mathrm{C})\cdot(1\ \mathrm{m/s})}{(1\ \mathrm{J})} \cdot \frac{(1\ e)}{(1.6\times 10^{-19}\ \mathrm{C})} \cdot \frac{(1\ c)}{(3\times 10^{8}\ \mathrm{m/s})} \cdot \frac{(10^{9}\ \mathrm{eV})(1.6\times 10^{-19}\ \mathrm{J})}{(1\ \mathrm{GeV})(1\ \mathrm{eV})}$$

$$= 3.34\ \frac{\mathrm{T}\cdot\mathrm{m}\cdot e}{\mathrm{GeV}/c}$$

$$(B\rho) = \left[3.34 \frac{(\mathrm{T}\cdot\mathrm{m})\, e}{(\mathrm{GeV}/c)}\right] \frac{p}{q}$$

Useful for momenta in GeV/*c* and charge in units of the elementary charge *e*

Example: a 1-GeV/*c* electron has a magnetic rigidity of 3.34 T m

Questions?

Magnetic fields in matter

Substances are divided into **3 main groups** according to their **behavior in magnetic fields**

DIAMAGNETIC

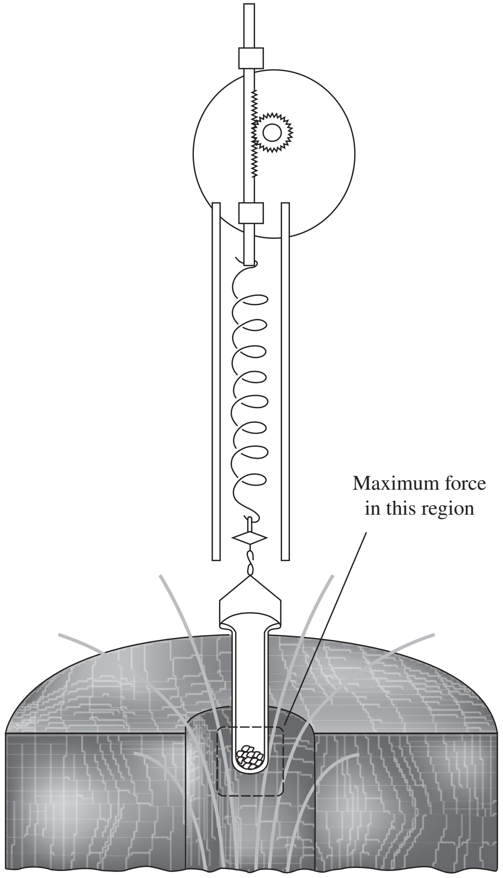
weak repulsion

PARAMAGNETIC

weak attraction

FERROMAGNETIC

very strong attraction



In accelerators, **ferromagnetic materials** are used in **electromagnets** to **amplify the magnetic field** by a factor 100-1000

Currents, magnetization and fields

Ampère's law in terms of **free** ("conduction") and **bound** ("microscopic") **currents**

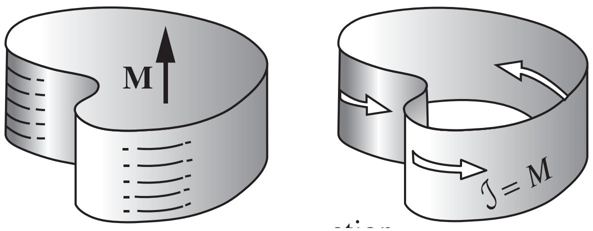
$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j}_f + \mathbf{j}_b \right)$$

Magnetization $\mathbf{M}$ = magnetic dipole density

related to bound currents [A m² / m³ = A / m]

Definition of $\mathbf{H}$ "magnetic field"

$$\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M}$$



$$\nabla \times \mathbf{M} = \mathbf{j}_b$$

$$\mathcal{J} = \mathbf{M} \times \hat{\mathbf{n}}$$

Ampère's law for free ("conduction") currents,
which are experimentally controllable

$$\nabla \times \mathbf{H} = \mathbf{j}_f \qquad \oint \mathbf{H} \cdot \hat{\mathbf{t}}\, ds = i_f$$

Useful to estimate magnet strengths

Magnetic properties of materials

Properties of materials ("equation of state") parameterized by

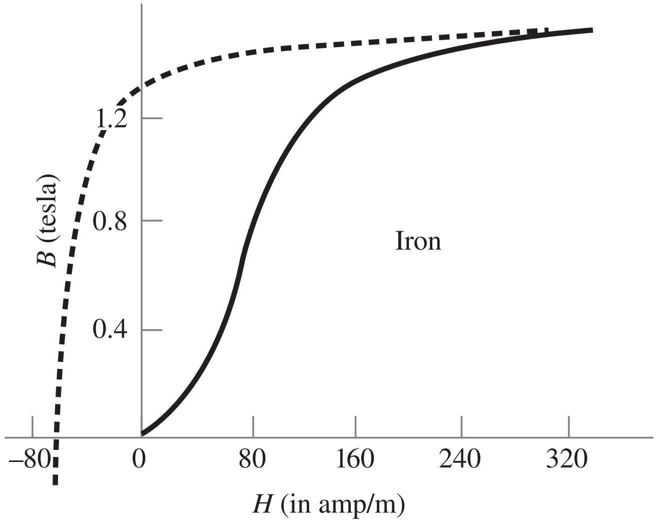
magnetic susceptibility χ_m

$$\mathbf{M} = \chi_m \mathbf{H}$$

magnetic permeability $\mu \equiv \mu_0(1 + \chi_m)$

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} \equiv \mu\mathbf{H}$$

Usually not a constant, especially for ferromagnets

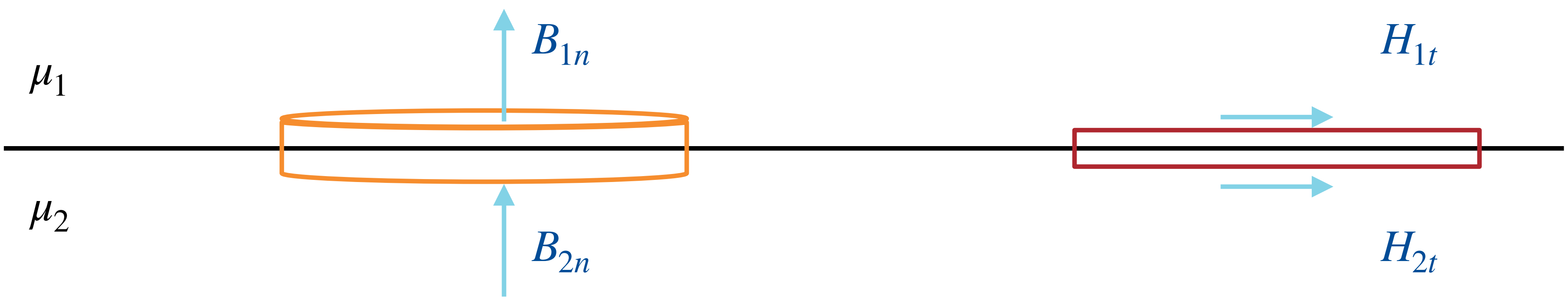


Hysteresis cycles are needed for reproducibility of magnet strengths

Boundary conditions at material interfaces

The surface integral of **B** is zero (always)

The line integral of **H** is zero (no conduction currents)



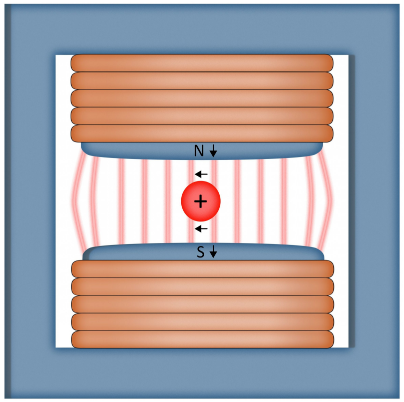
$$\nabla \cdot \mathbf{B} = 0 \implies \begin{cases} B_{1n} = B_{2n} \\ \mu_1 H_{1n} = \mu_2 H_{2n} \end{cases}$$

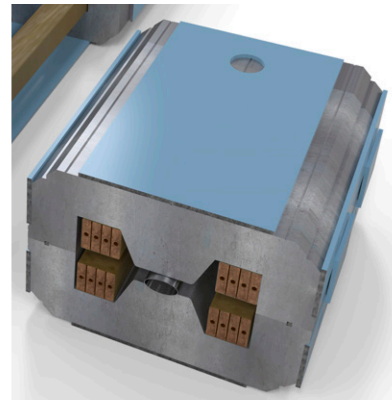
$$\nabla \times \mathbf{H} = 0 \implies \begin{cases} H_{1t} = H_{2t} \\ B_{1t}/\mu_1 = B_{2t}/\mu_2 \end{cases}$$

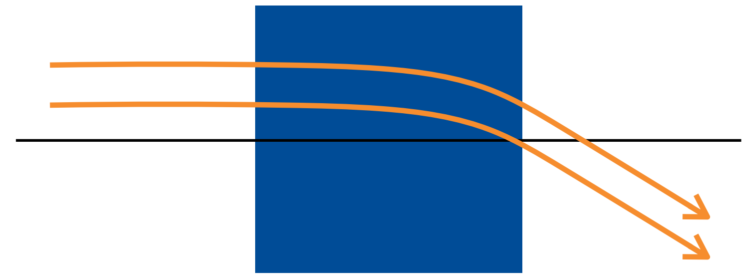
The **normal component** of the ***B*** field and
the **tangential component** of the ***H*** field
do not change

Main functions of electromagnets in accelerators

Dipole or “magnetic prism” for **deflection**

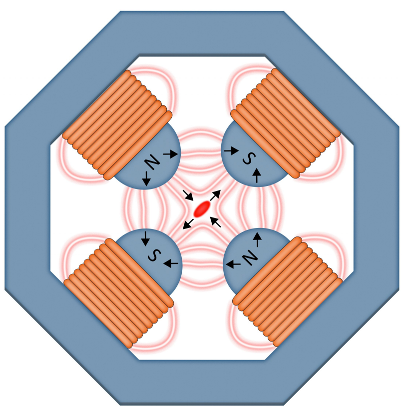


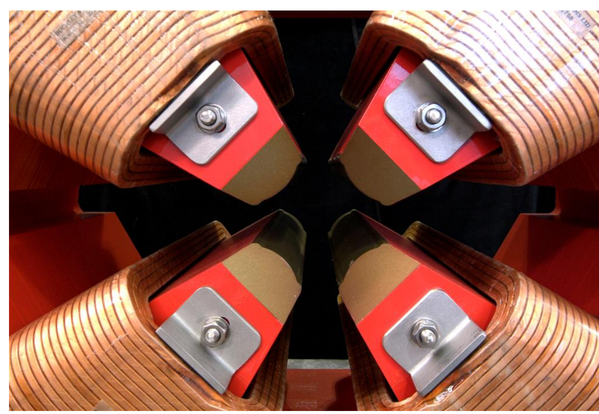


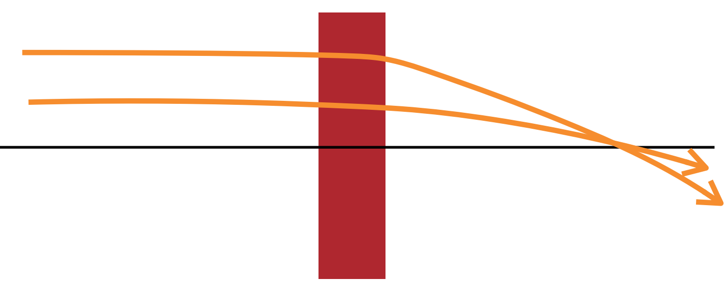


constant field

Quadrupole or “magnetic lens” for **focusing and defocusing**





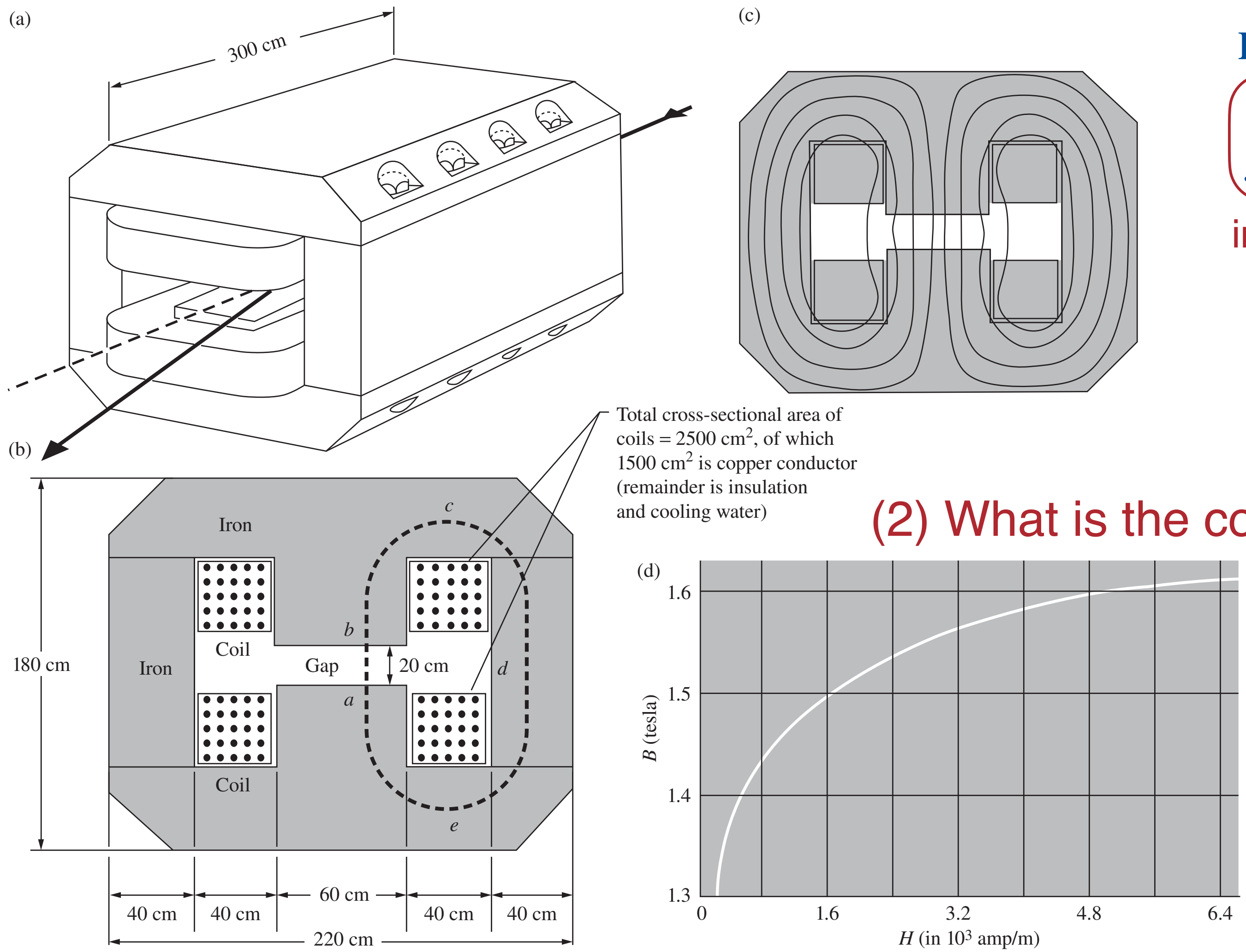


field proportional to distance from axis

plus many other types…

Basic dipole magnet design

(1) What ***B*** field is necessary to obtain the required deflection or "kick"?



$$\mathbf{F} = \dot{\mathbf{p}} \implies$$

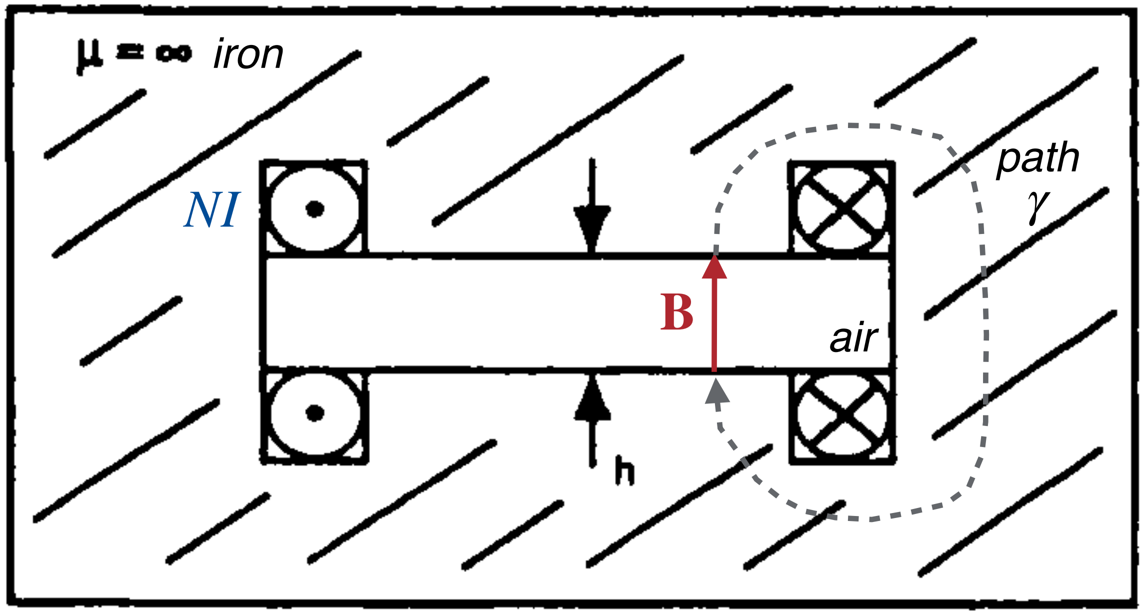
$$\int_{t_1}^{t_2} \mathbf{F}\, dt = \int_{s_1}^{s_2} \mathbf{F} \frac{ds}{v} = \mathbf{p}_2 - \mathbf{p}_1$$

impulse

(2) What is the corresponding ***H*** field?

(3) What current ***I*** is required to generate a given ***H***?

Dipole magnet strength



Show that the magnetic field is approximately

$$B = \frac{2\mu_0 NI}{h}$$

current in the coils

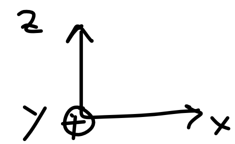
gap size

What assumptions did we make?

(Edwards and Syphers, problem 1.11)

Dipole magnet strength

Choice of coordinates



Use Ampère's law $\oint_\gamma \bar{H} \cdot \hat{t} \, ds = i_c$

tangent unit vector

Path γ clockwise in (x, z) plane $\Rightarrow$ surface normal unit vector $\hat{n} = \hat{y}$ $\Rightarrow$ $i_c = + 2NI$

Split the line integral $\oint = \int_{gap} + \int_{iron}$

In air, $\bar{H} = \bar{B}/\mu_0$; in iron $\bar{H} = \bar{B}/\mu$

Tangent vector in the gap $\hat{t} = \hat{z}$

$$\oint \bar{H} \cdot \hat{t} \, ds = \int_0^h \frac{B}{\mu_0} \hat{z} \cdot \hat{z} \, ds + \int_{iron} \frac{\bar{B}}{\mu} \cdot \hat{t} \, ds$$

negligible if $\mu \gg \mu_0$

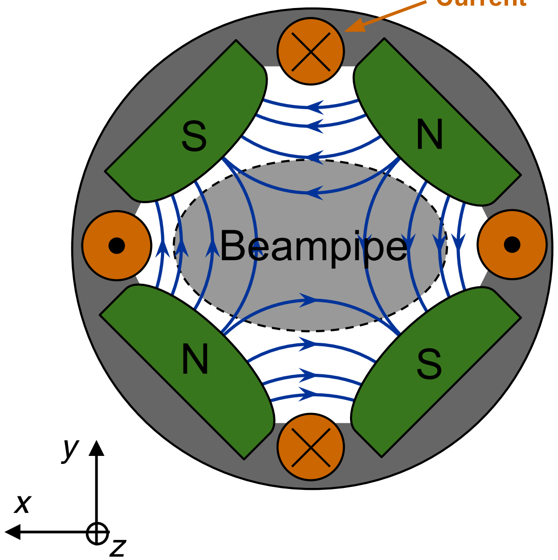
$$\simeq \frac{Bh}{\mu_0}$$

Therefore

$$B \simeq \frac{2 \mu_0 N I}{h}$$

Quadrupole magnet design

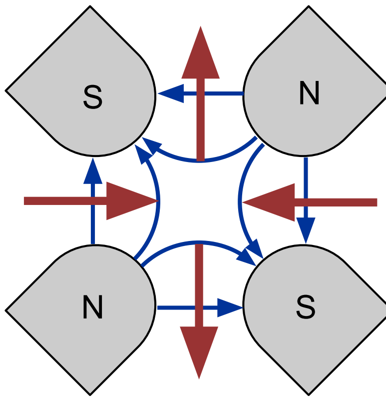
For an ideal quadrupole, the force is proportional to the distance from the axis



$$\mathbf{B}(x,y) = \begin{cases} B_x = B'y \\ B_y = B'x \end{cases}$$

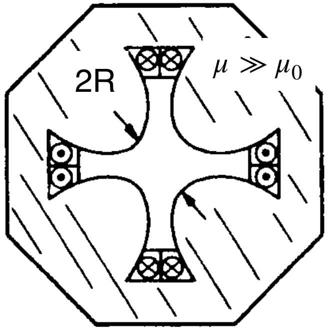
$$B = \sqrt{B_x^2 + B_y^2} = B'r$$

Force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$



Field **gradient** $B' > 0$ for a "focusing" or F magnet (positive particles moving into the page)
horizontal focusing and vertical defocusing

The quadrupole gradient is approximately



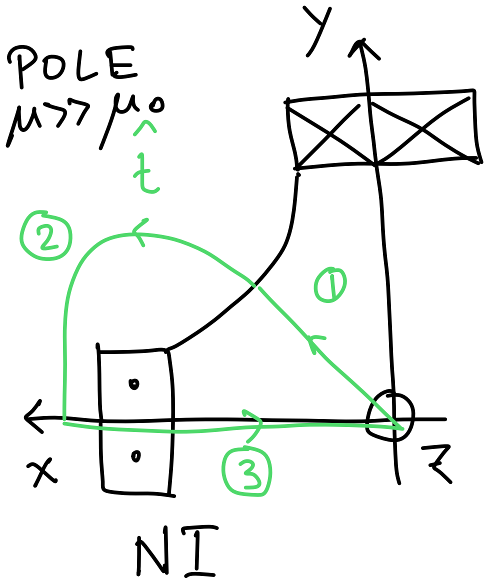
$$B' = \frac{2\mu_0 NI}{R^2}$$

current in the coils

square of the aperture radius

Fermilab

Quadrupole magnet strength



Counterclockwise path γ in the (x,y) plane. Surface normal unit vector $\hat{n} = -\hat{z} \parallel \bar{j} \Rightarrow i_c = +NI$

Split the path $\oint = \int_1 + \int_2 + \int_3$

$\int_2$: negligible $(\bar{H} = \bar{B}/\mu)$ $\quad$ $\int_3 = 0$ $(\bar{B} \perp \hat{t})$

Apply Ampère's law:

$$\oint \bar{H} \cdot \hat{t}\, ds = \int_0^R \frac{\bar{B}}{\mu_0} \cdot \hat{r}\, dr = \int_0^R \frac{B' r}{\mu_0}\, dr = \frac{B' R^2}{2\mu_0}$$

Therefore

$$B' = \frac{2\mu_0 NI}{R^2}$$

(see also Edwards and Syphers, problem 1.12)

Questions?