Introduction to Beam Physics and Accelerator Technology

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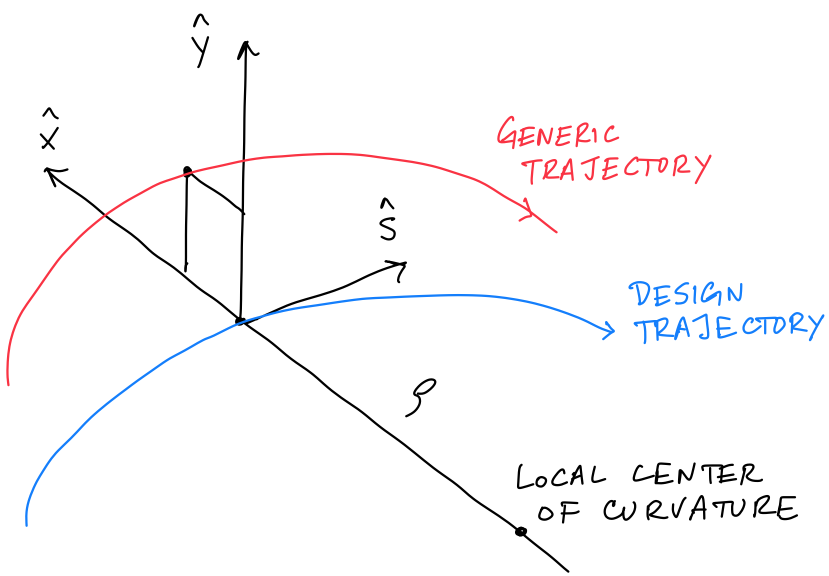
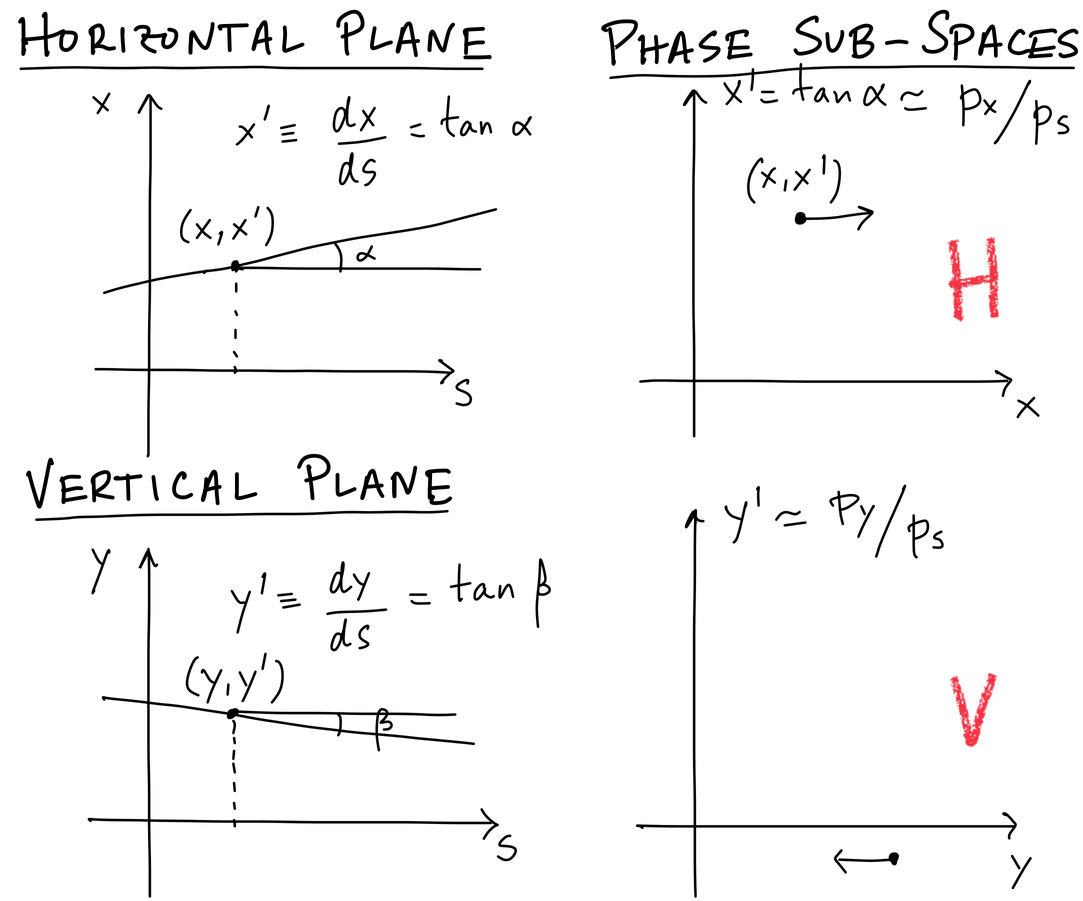
Transverse dynamics and focusing

Motion in the transverse plane

We study the motion of a charged particle in the electromagnetic fields of an accelerator

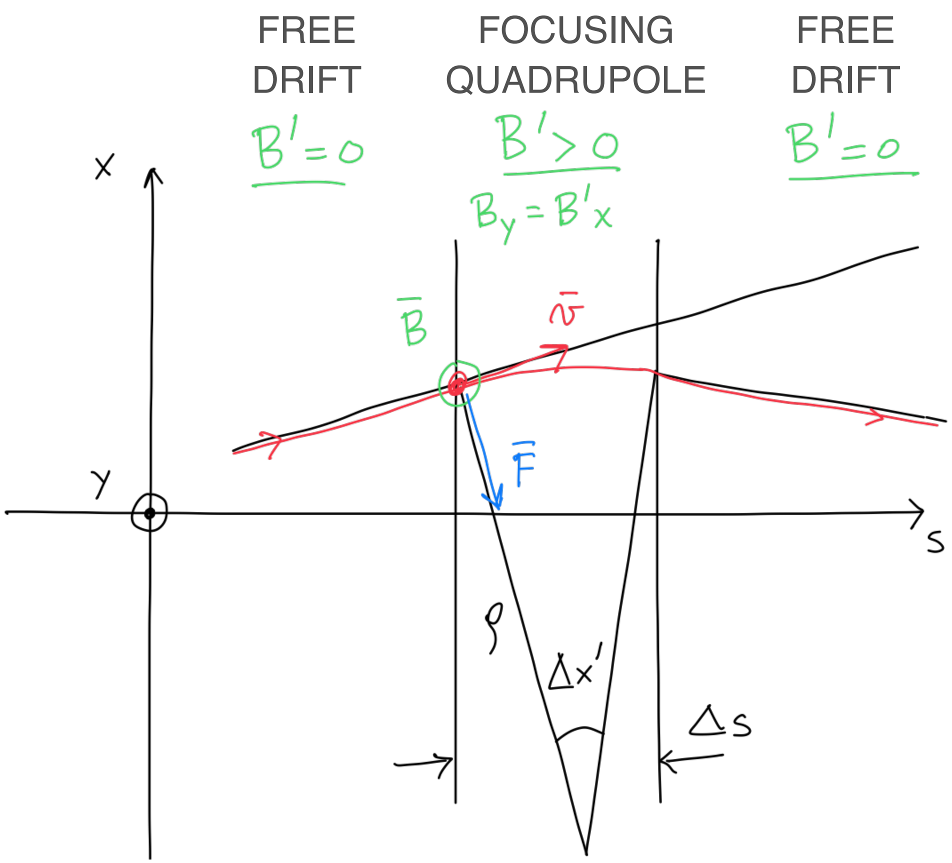
Choice of coordinates





Motion in a thin quadrupole region

The effects of magnetic fields can be analyzed as the superposition of components: dipole, quadrupole, etc. The quadrupole is the most basic focusing element.



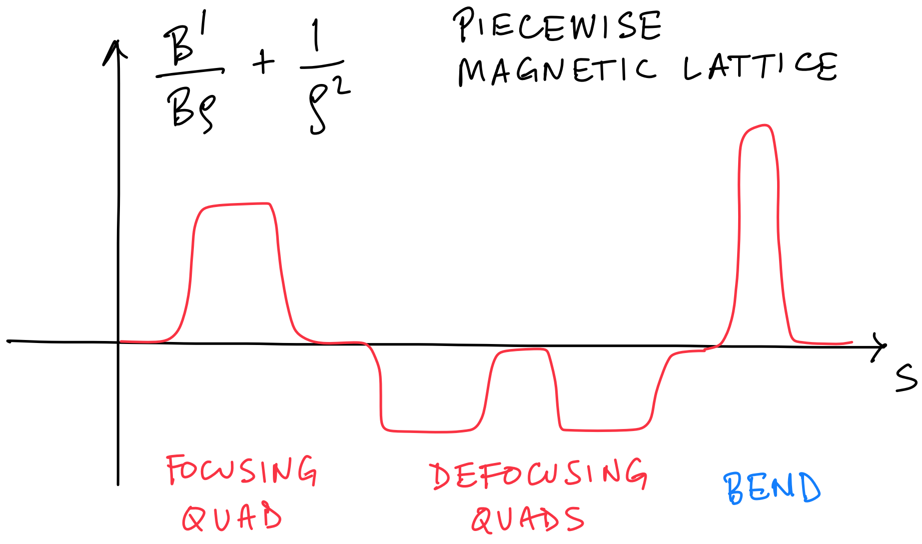
$$\Delta x' = -\frac{(\text{arc})}{\rho} \simeq -\frac{B_y \, \Delta s}{(B\rho)} = -\frac{B' \, \Delta s}{(B\rho)} x \equiv -\frac{x}{f}$$

quadrupole focusing power

angular kick proportional to orbit deviation (like a thin lens in optics)

Piecewise distribution of gradients: equations of motion

We consider a beam line or a ring with a **distribution of gradients** along its length



equations of motion

$$\Delta x' = -\frac{B' \Delta s}{(B\rho)} x \implies x'' + \frac{B'}{(B\rho)} x = 0$$

normalized gradient *[length⁻²]*

$$y'' + \frac{(-B')}{(B\rho)} y = 0$$

If there is a **curvature** (**dipoles**), the complete equation includes a **centripetal term**

$$x'' + \left[\frac{B'}{(B\rho)} + \frac{1}{\rho^2}\right] x = 0$$

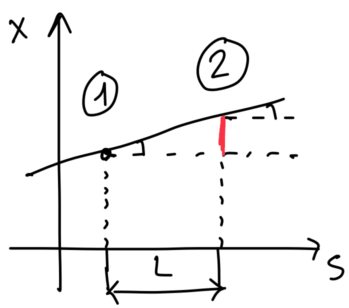
Hill's equation: similar to **harmonic oscillator**, but **with variable restoring force**

Discrete description of transverse motion: transport matrices

For the **design** of accelerator systems and to study their **stability**, it is convenient to introduce **transport matrices**, which **transform the phase-space "vectors"** (x, x') and (y, y') or $(x, x', y, y', z, \delta_p)$

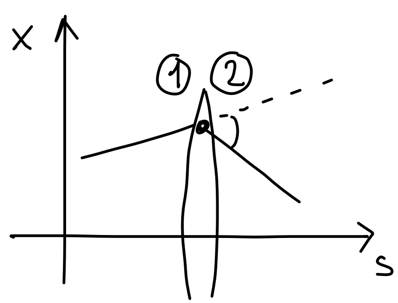
$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

free **DRIFT** of length *L*



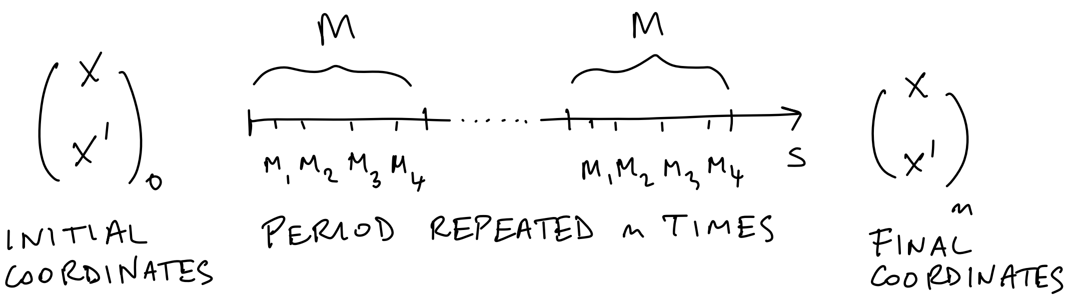
$$\begin{cases} x_2 = x_1 + L x_1' \\ x_2' = x_1' \end{cases} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad \det M = 1$$

THIN LENS of focal length *f*



$$\begin{cases} x_2 = x_1 \\ x_2' = x_1' - \dfrac{x_1}{f} \end{cases} \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_1 \qquad \det M = 1$$

Stability of a periodic focusing system



$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M \cdot M \cdot \ldots \cdot M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Do the coordinates **diverge**?

Definitions

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Eigenvalues $\lambda_1 \equiv e^{i\mu}$ and $\lambda_2 \equiv e^{-i\mu}$

$$\det M = 1 \implies \lambda_2 = 1/\lambda_1$$

Initial vector as linear combination of eigenvectors v_1 and v_2 of M

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = A v_1 + B v_2 \implies \begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0 = A \lambda_1^n v_1 + B \lambda_2^n v_2$$

For **stability**, we **require** that the λ_i^n do not diverge and that μ **be real**

Stability condition for a periodic system

Eigenvalue equation for M

$$\det(M - \lambda I) = 0 \qquad \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \qquad (a-\lambda)(d-\lambda) - bc = 0$$

$$(ad - bc) - \lambda(a+d) + \lambda^2 = 0 \qquad \lambda + \frac{1}{\lambda} = a + d \qquad e^{i\mu} + e^{-i\mu} = \operatorname{tr} M \qquad 2\cos\mu = \operatorname{tr} M$$

$(ad - bc) = \det M = 1$

$$\boxed{-2 \leq \operatorname{tr} M \leq 2}$$

Stability condition for the transport matrix of a periodic system

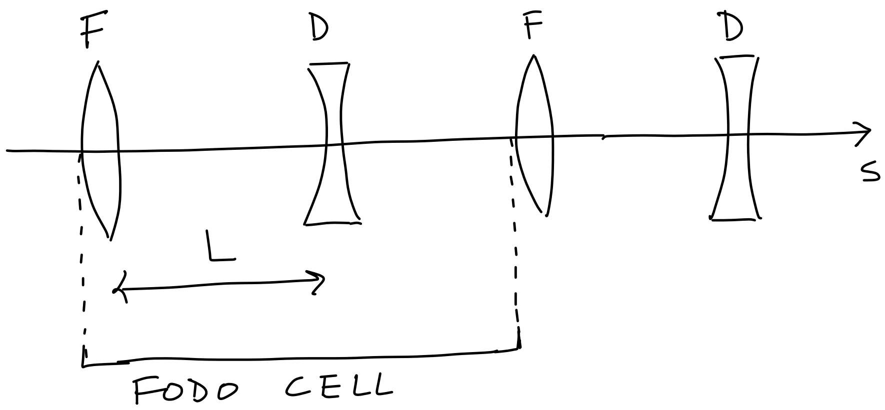
The parameter μ has a physical interpretation: **phase advance** of the transverse oscillation in one period

The trace is invariant under cyclic permutations: the stability condition does not depend on the starting point

Example: properties of the FODO cell

Consider a **periodic system** consisting of a particular case of **alternating gradients**:

- a thin focusing lens with focal length *f*
- a drift space of length *L*
- a thin defocusing lens with focal length *-f*
- another drift space of length *L*



pay attention to the order of the matrices

$$M = M_O \cdot M_D \cdot M_O \cdot M_F$$

(a) Calculate the transport matrix of the FODO cell. Verify that its determinant is 1.

(b) Find the relation that *f* and *L* must satisfy for stability

(c) Find an expression for the phase advance

(d) What is the transport matrix of the DOFO cell? Does it have the same trace?

Example: properties of the FODO cell

$$M_{\text{FODO}} = \begin{pmatrix} 1 - L/f - L^2/f^2 & 2L + L^2/f \\ -L/f^2 & 1 + L/f \end{pmatrix}$$

$$\operatorname{tr} M_{\text{FODO}} = 2 - \frac{L^2}{f^2} \qquad \cos\mu = 1 - \frac{L^2}{2f^2}$$

For stability: $L \leq 2\,|\,f\,|$

Alternating gradients are stable as long as the distance between lenses does not exceed twice the focal length

Linearized equations of motion and Courant-Snyder parameters

$$\begin{cases} x'' + \left[\dfrac{B'}{(B\rho)} + \dfrac{1}{\rho^2}\right] x = 0 \\ y'' + \dfrac{(-B')}{(B\rho)} y = 0 \end{cases}$$

$$x'' + K(s) \cdot x = 0$$

In circular machines and other periodic systems, the **normalized gradient** K is periodic

$$K(s + L) = K(s)$$

Solutions of Hill's equation

constants determined by the initial conditions

$$x(s) = \sqrt{\beta(s) \cdot \epsilon} \cdot \cos\left[\psi(s) + \delta\right]$$

beta(tron) function
depends on $K(s)$ and has the same periodicity

phase advance
in general not linear in s

single-particle emittance

Phase advances and betatron tunes

The functions $\beta(s)$ and $\psi(s)$ are not independent $\psi = \int \frac{1}{\beta} ds$

The phase advance between two points is $\Delta\psi = \psi_2 - \psi_1 = \int_{s_1}^{s_2} \frac{1}{\beta} ds$

The number of oscillations in one period is called **betatron tune**

$$\nu = \frac{1}{2\pi} \oint \frac{1}{\beta} ds$$

Stability of the beam in a real machine is **very sensitive** to the value of the betatron tune. It is **one of the most important accelerator parameters**.

Courant-Snyder parameters and phase space

$\beta(s)$ — **beta(tron) function** or amplitude function [m]

$$\alpha(s) \equiv -\frac{\beta'(s)}{2}$$ [adimensional]

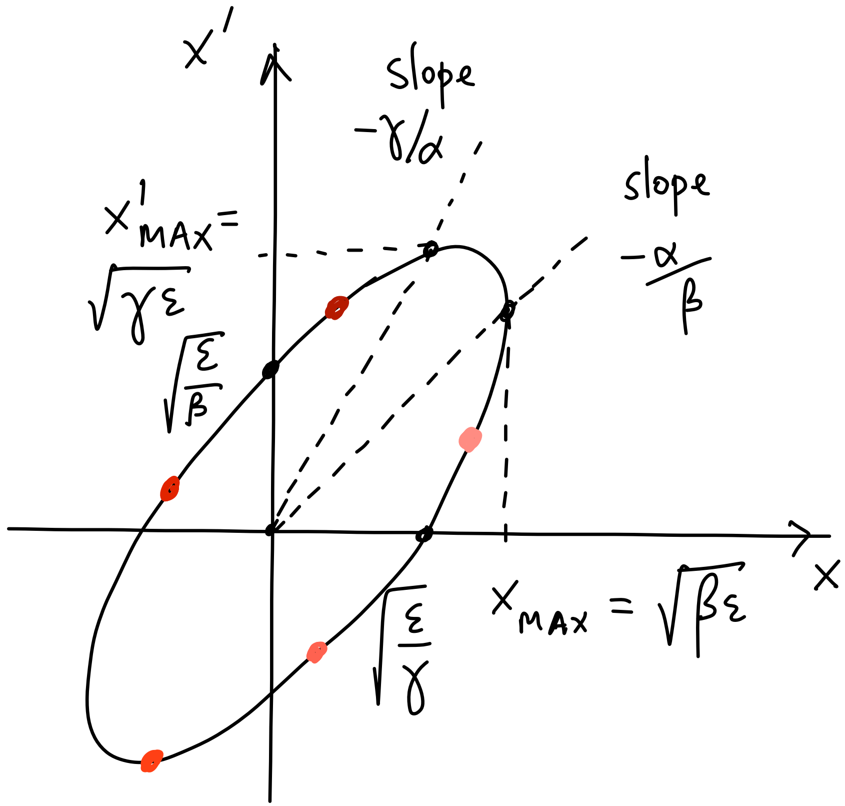
$$\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$$ [m^{-1}]

Useful to
describe particle motion
express the elements of transport matrices

Courant-Snyder parameters and phase space

From the solutions of Hill's equation one can show that $\gamma x^2 + 2\alpha x x' + \beta (x')^2 = \epsilon$

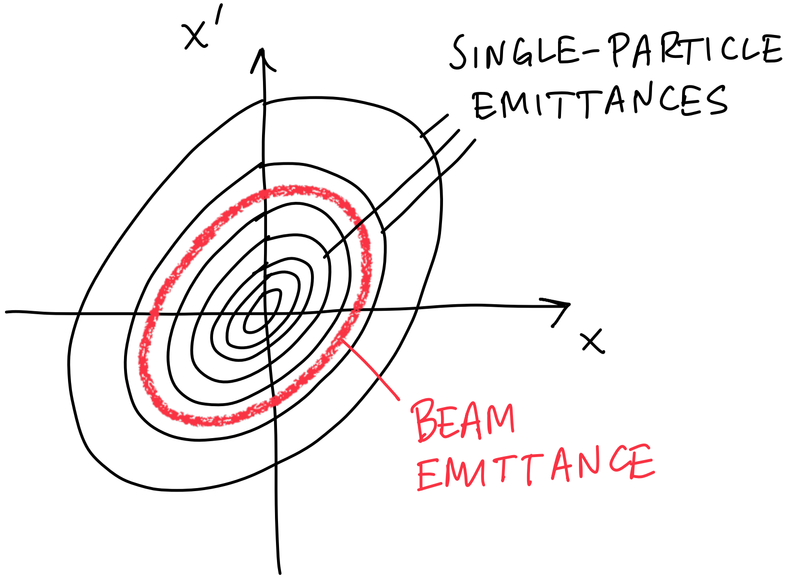
For each position s along the accelerator, there is an ellipse in phase space (x, x') that describes particle motion



The **location** of the particle on the ellipse is determined by the **initial phase**, by the **betatron tune** and by the **turn number**

Beam emittance

The **beam emittance** is the emittance that contains a specified fraction of the single-particle beam emittances



For Gaussian beams, the "rms" (39%) emittances ϵ_x and ϵ_y are related to the **rms beam sizes**

$$\epsilon_x = \frac{\sigma_x^2}{\beta_x} \qquad \epsilon_y = \frac{\sigma_y^2}{\beta_y}$$

Courant-Snyder parameters and transport matrices

Transport matrix of **one period**

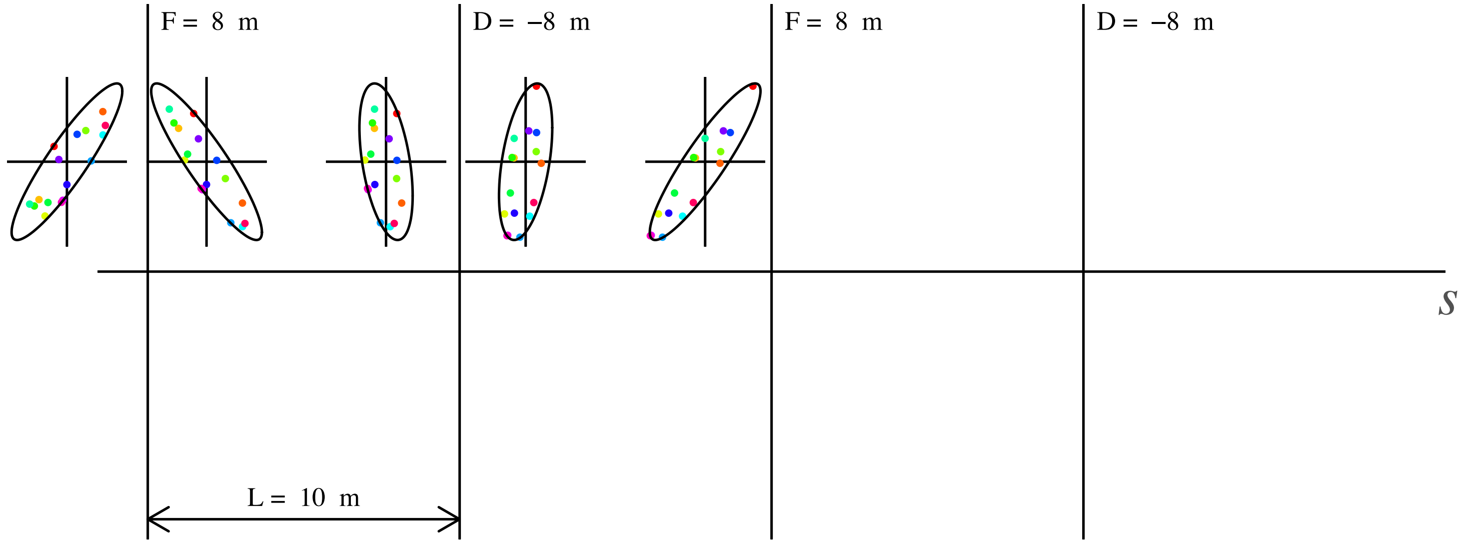
$$M = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$

Transport matrix **between two locations**

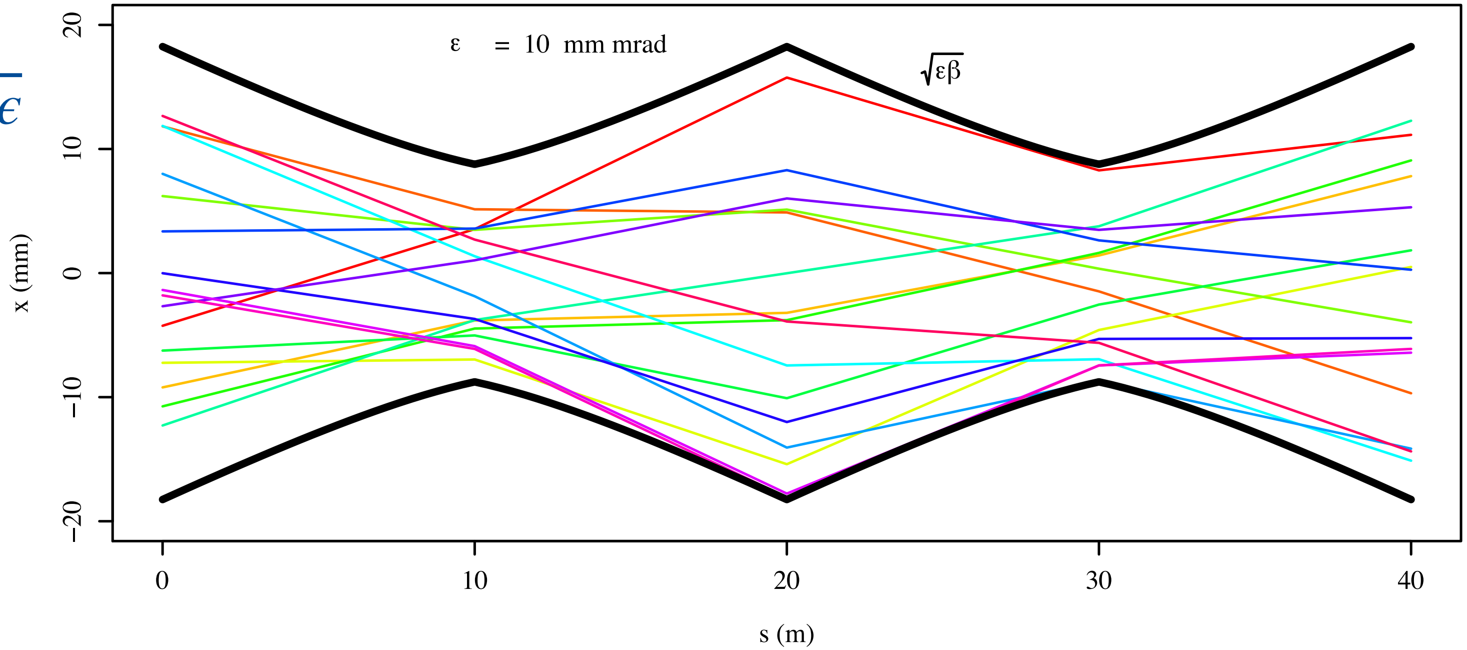
$$M(s_1 \to s_2) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\psi + \alpha_1\sin\Delta\psi) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{pmatrix}$$

Example: evolution of the beam in the FODO cell

Evolution of the (x, x') ellipse



Evolution of the beam envelope $\sqrt{\beta\epsilon}$



Deviations from ideal linear motion

Imperfections in the fields create **resonances**, which are amplified if tunes are close to rational numbers

$$m_x \cdot \nu_x + m_y \cdot \nu_y = m \quad \text{with integer } m_x, m_y, m$$

The distribution of particle momenta generates differences in focusing, parameterized by **chromaticity** ξ

$$\Delta\nu = \xi \frac{\Delta p}{p}$$

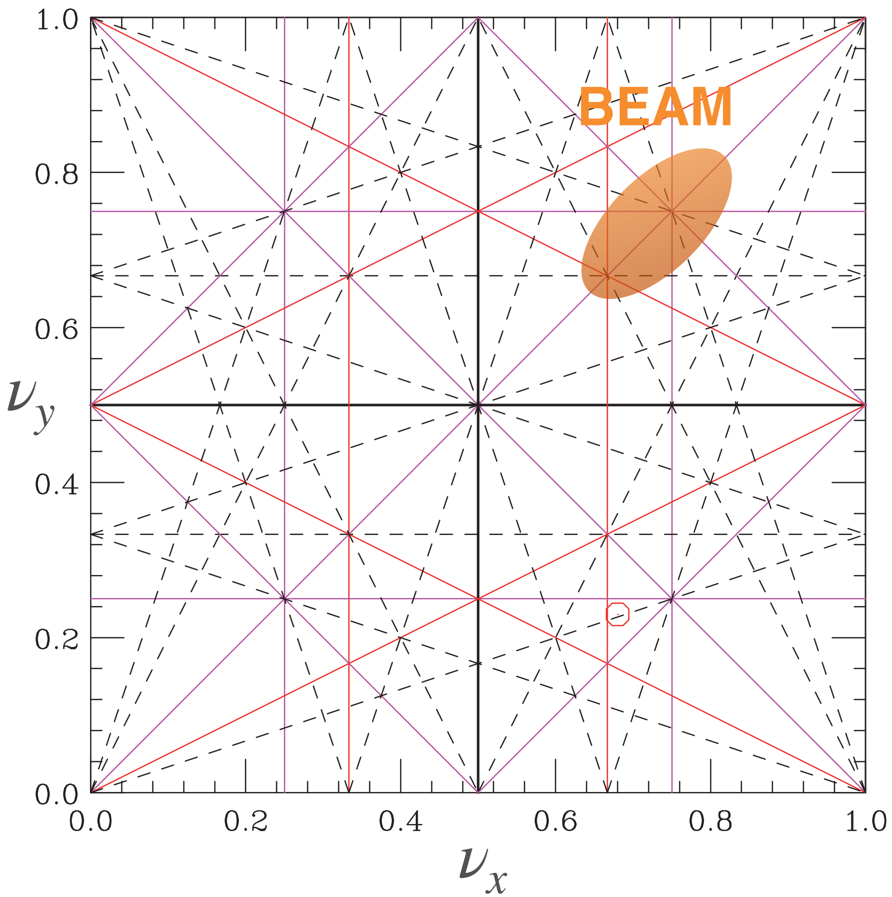
Intense beams experience **self fields** and **wake fields** in addition to the external focusing fields

In colliders, the **beam-beam force** between colliding bunches is intense and highly nonlinear

Studying the rich **interplay** between these **complex effects** is stimulating and challenging. Understanding of phenomena yields advances, solutions and, often, more questions!

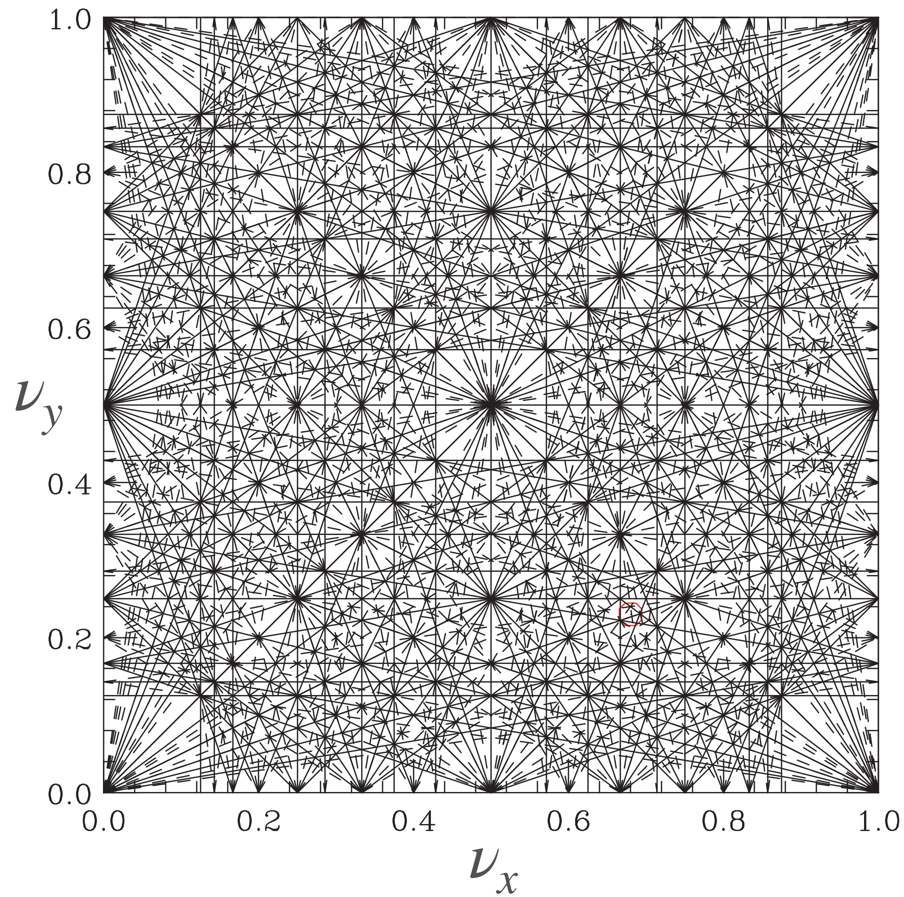
Beam distributions in the tune diagram

One way to visualize the effects on dynamics is through the **tune diagram**



$$m_x \cdot \nu_x + m_y \cdot \nu_y = m$$

$$|m_x| + |m_y| \leq 4$$



$$m_x \cdot \nu_x + m_y \cdot \nu_y = m$$

$$|m_x| + |m_y| \leq 8$$

Concepts in nonlinear dynamics

Nonlinear dynamics is a vast field with many applications in physics, engineering, biology, etc.

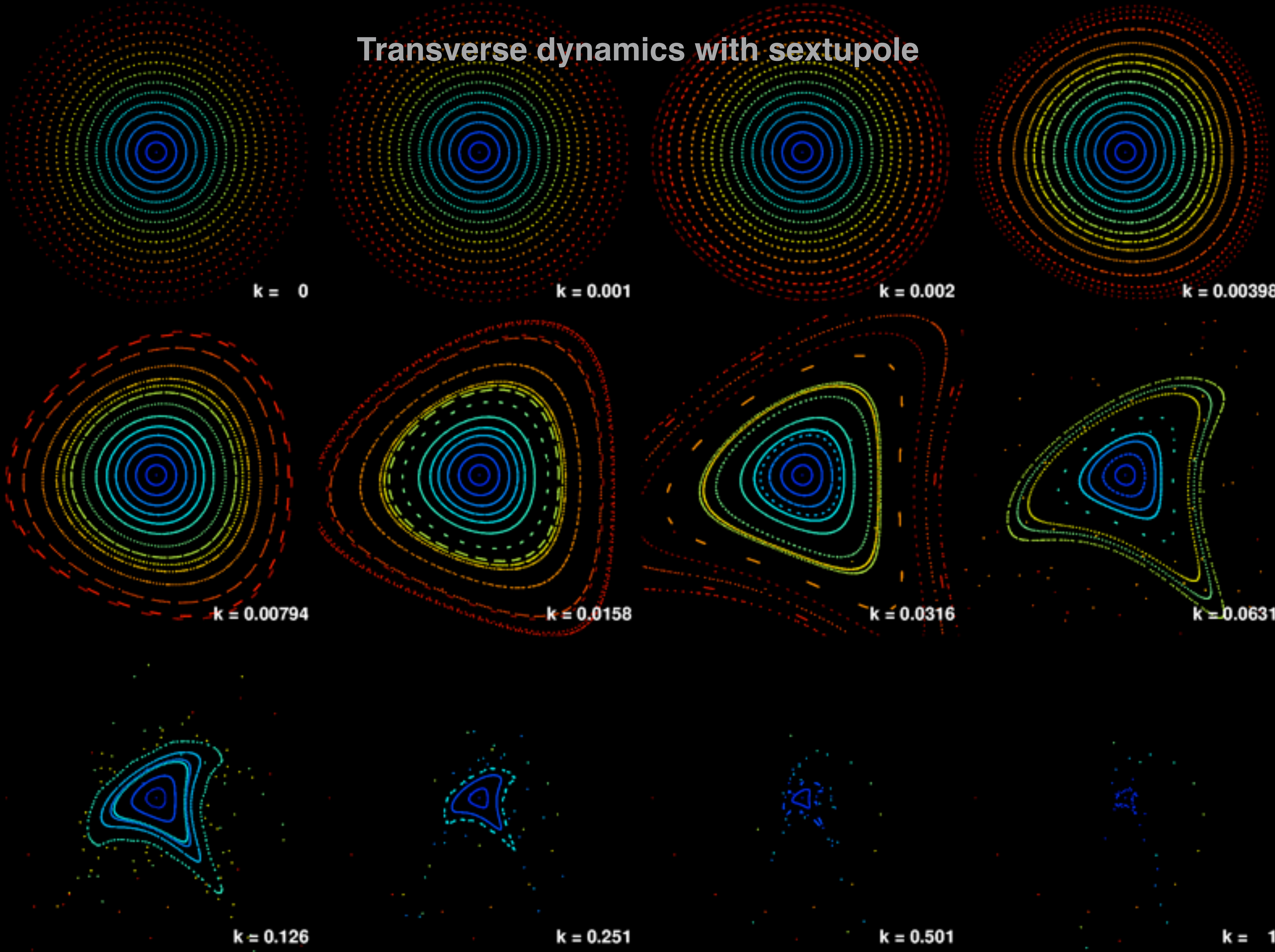
In phase space, **nonlinear equations** of motion generate **regular and chaotic regions**

Deterministic **chaos** manifests itself as an extremely **sensitive dependence on initial conditions** and in motion that **appears random**

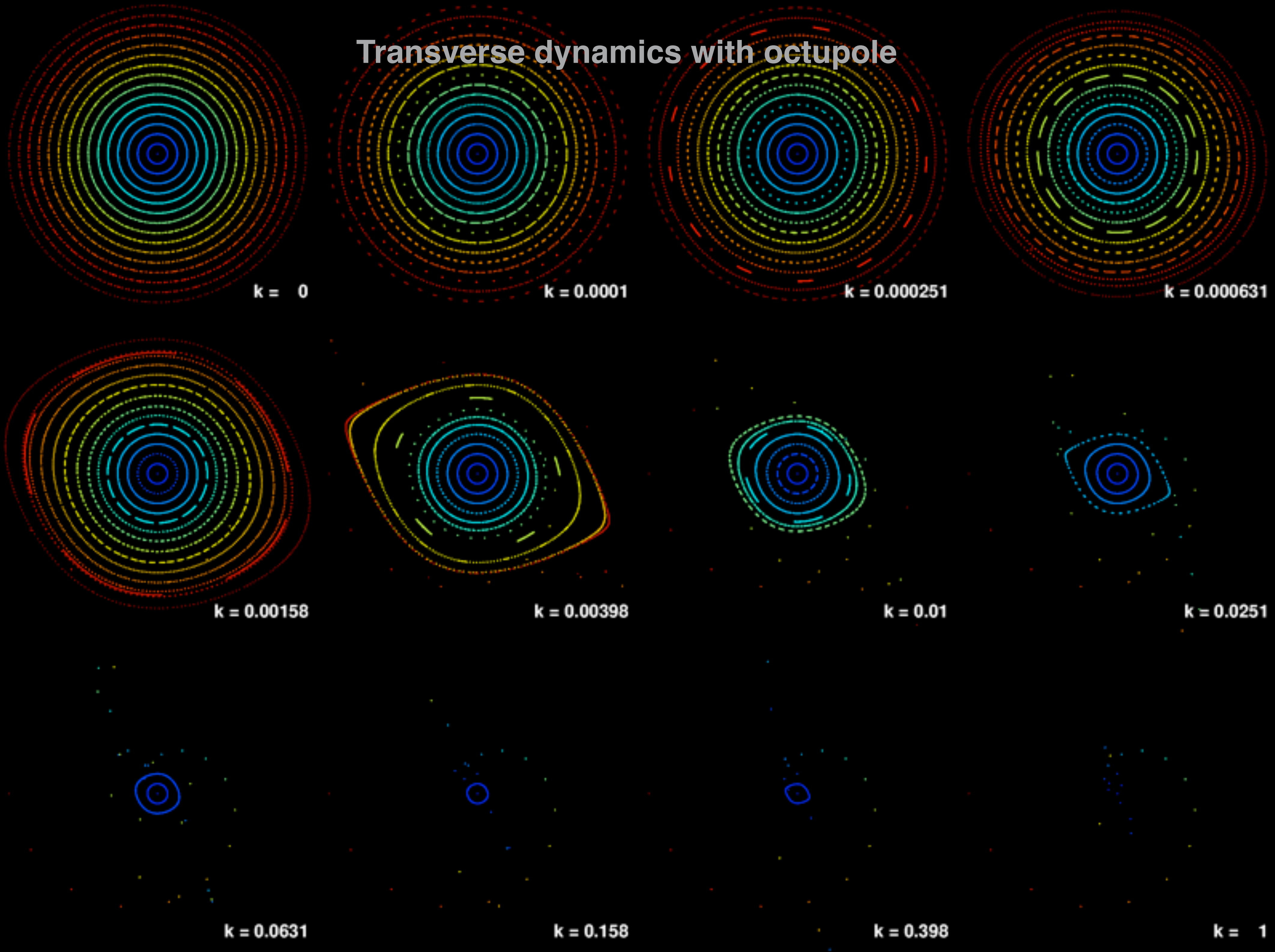
When motion is nonlinear and periodic, oscillation **frequencies** in general **depend on amplitude**

In accelerator physics, the concept of **dynamic aperture** represents the region of phase space where motion is stable

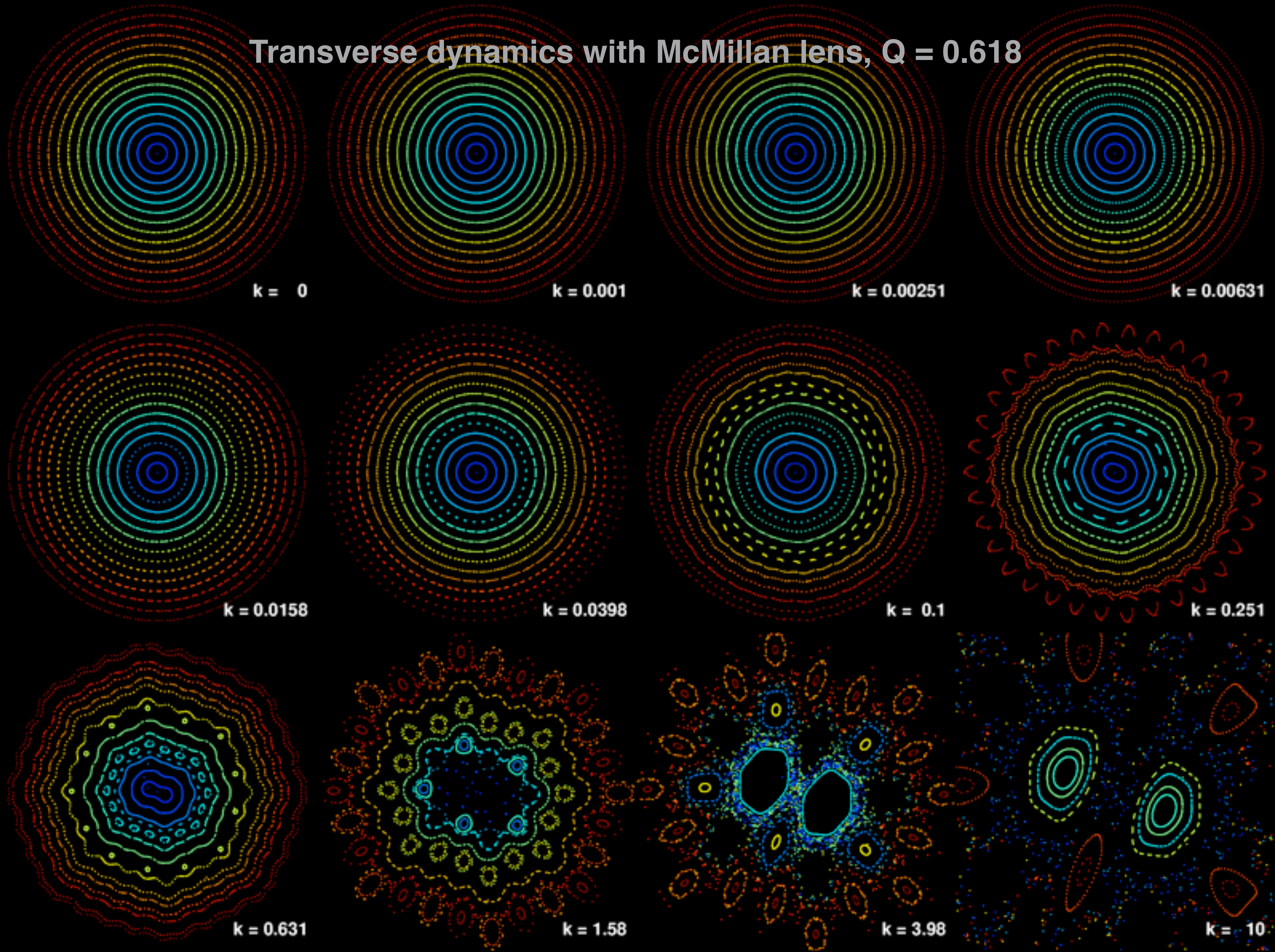
Transverse dynamics with sextupole



Transverse dynamics with octupole



Transverse dynamics with McMillan lens, Q = 0.618



Transverse dynamics with McMillan lens, Q = 0.25