Algebraic Techniques in Differential Cryptanalysis

. . . walking that extra mile with algebra

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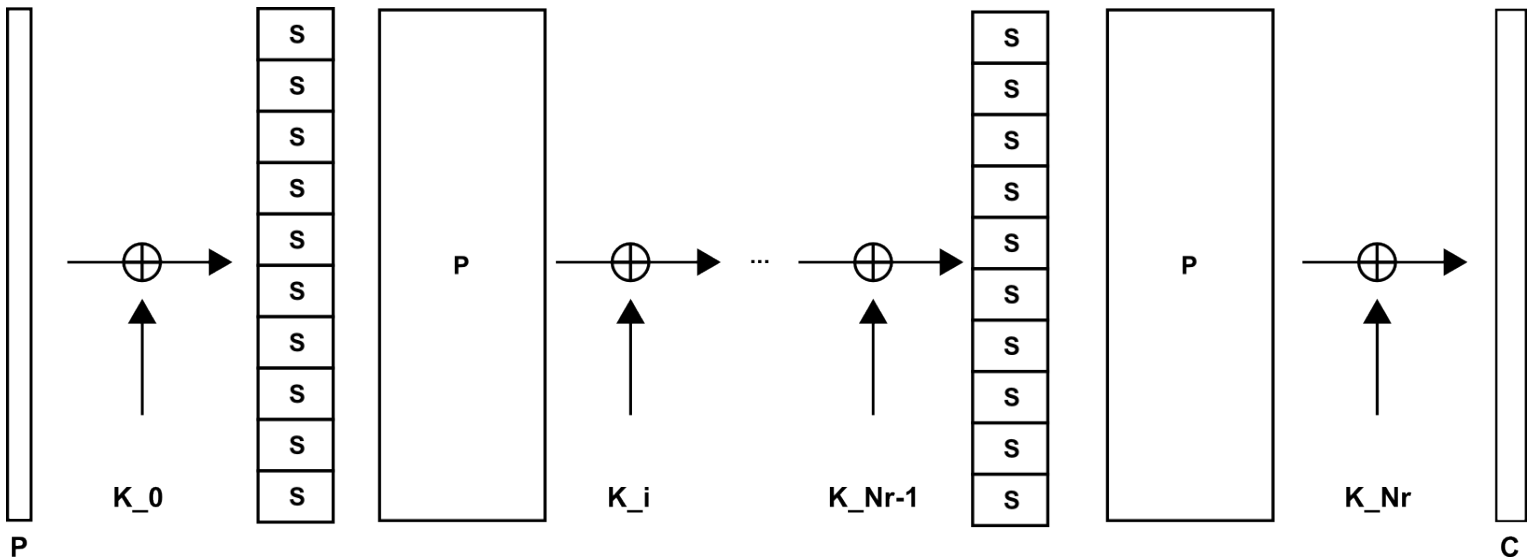
[1] joint work with Carlos Cid

Outline

Outline

The Blockcipher PRESENT

PRESENT [4] was proposed by Bogdanov et al. at CHES 2007 as an ultra-lightweight block cipher, suitable for RFIDs and similar devices.



Where the S-Box is defined as

$$S = [12, 5, 6, 11, 9, 0, 10, 13, 3, 14, 15, 8, 4, 7, 1, 2]$$

and the permutation layer P as $4 \cdot j + i \rightarrow 16 \cdot i + j$ with $(0 \leq j < 16, 0 \leq i < 4)$.

PRESENT-80 Key Schedule

The user-supplied key is stored in key register K and represented as

$$K = k_{79}k_{78}\ldots k_0.$$

At round i the round key K_i consists of the 64 most significant bits of K.

$$K_i = k_{i,63}k_{i,62}\ldots k_{i,0} = k_{79}k_{78}\ldots k_{16}.$$

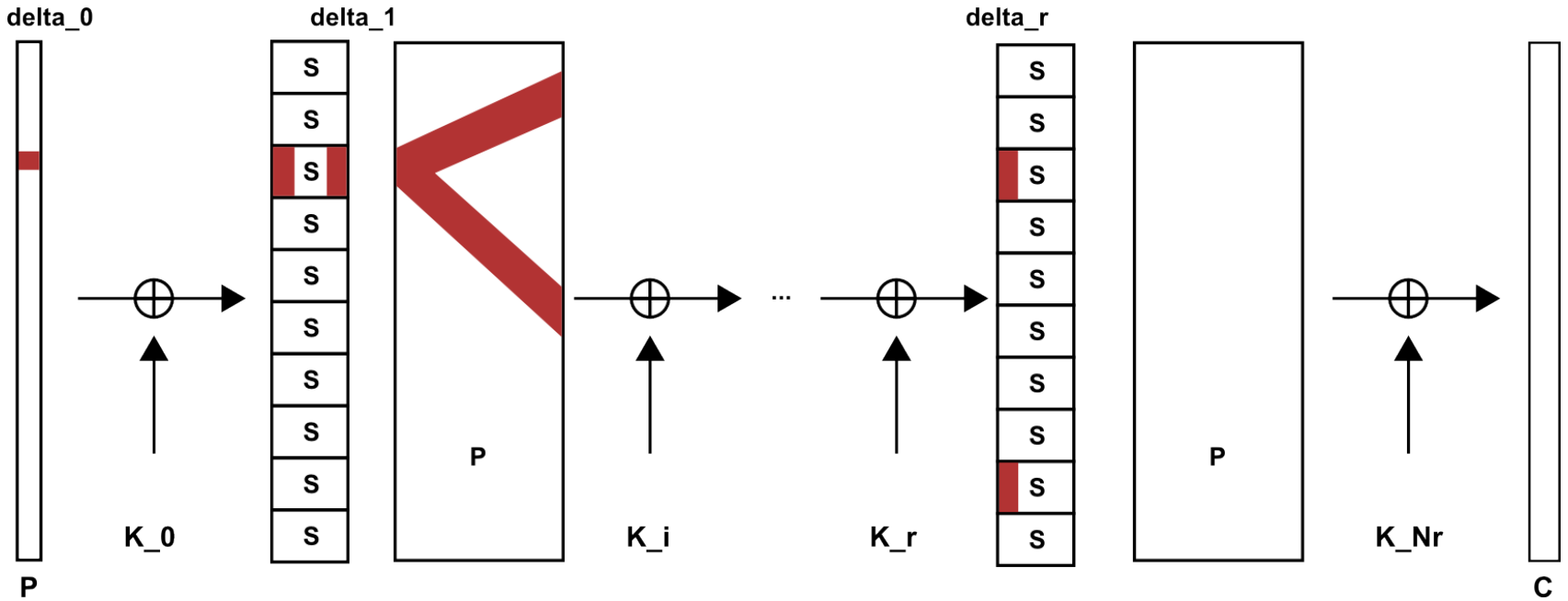
Afterwards, the key register is updated:

1. $[k_{79}k_{78}\ldots k_1k_0] = [k_{18}k_{17}\ldots k_{20}k_{19}]$
2. $[k_{79}k_{78}k_{77}k_{76}] = \mathbf{S}[\mathbf{k_{79}k_{78}k_{77}k_{76}}]$
3. $[k_{19}k_{18}k_{17}k_{16}k_{15}] = [k_{19}k_{18}k_{17}k_{16}k_{15}] \oplus \text{round\_counter}$

The key schedule for 128-bit keys is quite similar.

Outline

Differential Cryptanalysis I



$$Pr(\delta_i) = p_i \longrightarrow Pr(\Delta) = \prod p_i$$

Differential Cryptanalysis II

Key Recovery:

- **backward key guessing** to recover subkey bits of last rounds not covered by characteristic
- **right pairs** suggest correct and wrong key bits
- **wrong pairs** suggest random key bits
- **filter functions** used to remove wrong pairs
- **candidate key arrays** to count suggestions and observe peak

Differential Cryptanalysis III

Properties of the attack:

- One of the most successful attack techniques against block ciphers, hash functions, etc. [2].
- Usually requires huge quantities of plaintext–ciphertext pairs ($> 2^{-p}$).
- Attack is well understood, so modern block ciphers usually do not have their security affected.

Differential Cryptanalysis of 16-round DES [3]

- distinguishes right pairs,
- uses outer round active S-Boxes to recover key bits and
- does not rely on candidate key arrays.

Difference Distribution Matrix for PRESENT

$$
\begin{pmatrix}
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 4 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 2 & 0 \\
0 & 2 & 0 & 2 & 2 & 0 & 4 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 4 & 2 & 0 & 0 & 4 \\
0 & 4 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 4 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 4 \\
0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\
0 & 0 & 2 & 2 & 0 & 4 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 \\
0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 4 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 \\
0 & 2 & 4 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4
\end{pmatrix}
$$

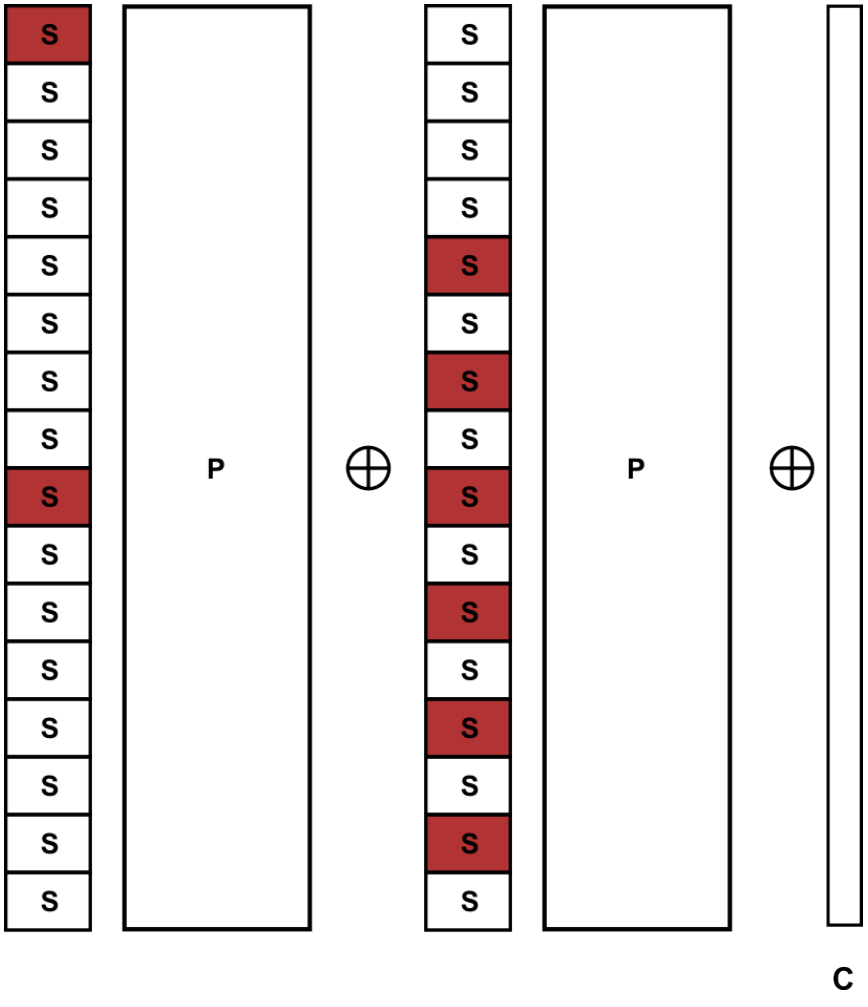
$$
Pr(\Delta X = 1 \rightarrow \Delta Y = 3) = \frac{4}{16} = \frac{1}{4}
$$

14-Round Characteristic for PRESENT [12]

Rounds		Differences	Pr	Rounds		Difference	Pr
I		$x_2 = 7, x_{14} = 7$	1				
R1	S	$x_2 = 1, x_{14} = 1$	2^{-4}	R8	S	$x_8 = 9, x_{10} = 9$	2^{-4}
R1	P	$x_0 = 4, x_3 = 4$	1	R8	P	$x_2 = 5, x_{14} = 5$	1
R2	S	$x_0 = 5, x_3 = 5$	2^{-4}	R9	S	$x_2 = 1, x_{14} = 1$	2^{-6}
R2	P	$x_0 = 9, x_8 = 9$	1	R9	P	$x_0 = 4, x_3 = 4$	1
R3	S	$x_0 = 4, x_8 = 4$	2^{-4}	R10	S	$x_0 = 5, x_3 = 5$	2^{-4}
R3	P	$x_8 = 1, x_{10} = 1$	1	R10	P	$x_0 = 9, x_8 = 9$	1
R4	S	$x_8 = 9, x_{10} = 9$	2^{-4}	R11	S	$x_0 = 4,\ x_8 = 4$	2^{-4}
R4	P	$x_2 = 5, x_{14} = 5$	1	R11	P	$x_8 = 1,\ x_{10} = 4$	1
R5	S	$x_2 = 1, x_{14} = 1$	2^{-6}	R12	S	$x_8 = 9,\ x_{10} = 9$	2^{-4}
R5	P	$x_0 = 4, x_3 = 4$	1	R12	P	$x_2 = 5,\ x_{14} = 5$	1
R6	S	$x_0 = 5, x_3 = 5$	2^{-4}	R13	S	$x_2 = 1,\ x_{14} = 1$	2^{-6}
R6	P	$x_0 = 9, x_8 = 9$	1	R13	P	$x_0 = 4,\ x_3 = 4$	1
R7	S	$x_0 = 4, x_8 = 4$	2^{-4}	R14	S	$x_0 = 5,\ x_3 = 5$	2^{-4}
R7	P	$x_8 = 1, x_{10} = 1$	1	R14	P	$x_0 = 9,\ x_8 = 9$	1

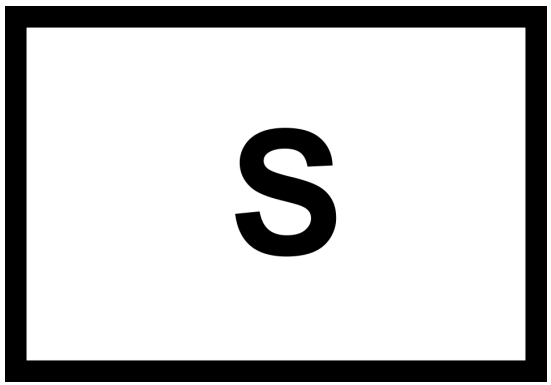
Table: 14-round differential characteristic for PRESENT with probability 2^{-62}

Two Round Filter Function for PRESENT-80-16 [12]



- Two S-Boxes are active in round 15.
- Each has six possible output differences.
- We have 36 possible output differences for round 15.
- At most 6 S-Boxes are active in round 16.
- We can discard $\sim 2^{50}$ out of 2^{62} pairs.

Algebraic Cryptanalysis I



$$
\begin{aligned}
&y_2x_3 + y_3x_3 + x_1x_3 + x_2x_3 + x_3,\\
&y_0x_3 + y_3x_3 + x_1x_3 + x_2x_3 + \ldots,\\
&x_1x_2 + y_3 + x_0 + x_1 + x_3,\\
&x_0x_2 + y_3x_3 + x_1x_3 + x_2x_3 + \ldots\\
&y_3x_2 + y_3x_3 + x_1x_3 + y_0 + y_1 + y_3 \ldots\\
&y_0x_2 + y_1x_2 + y_1x_3 + y_3x_3 + \ldots\\
&x_0x_1 + y_3x_3 + x_1x_3 + x_2x_3 + \ldots\\
&y_3x_1 + y_3x_3 + x_2x_3 + \ldots, \ldots
\end{aligned}
$$

We call $X_{i,j}$ and $Y_{i,j}$ the input resp. output variable for the j-th bit of the i-th S-Box application (i.e. round).

For PRESENT-80-31 we would have a system of 8140 variables in 34742 equations if we consider two plaintext-ciphertext pairs.

Algebraic Cryptanalysis II

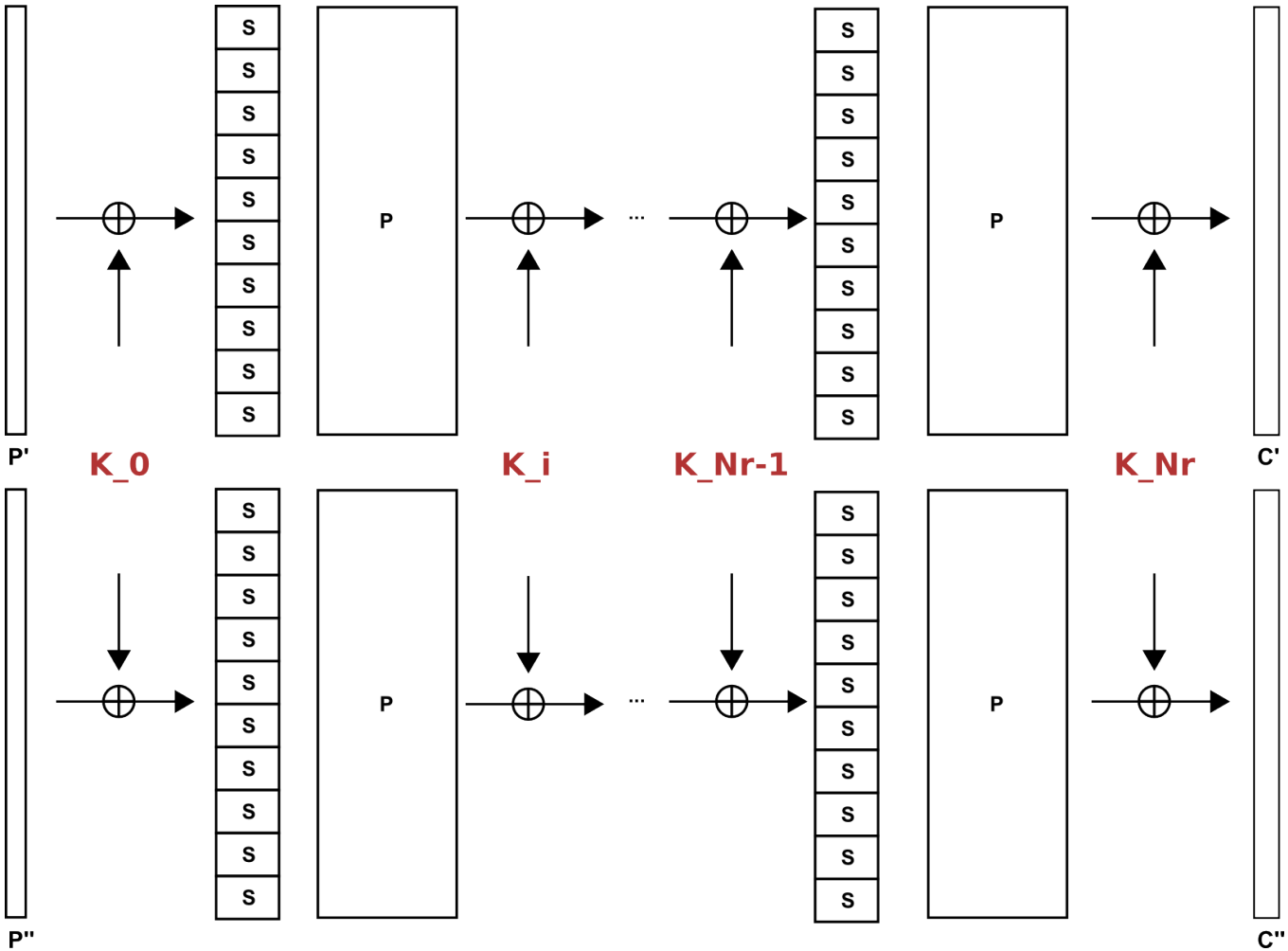
Properties of the attack:

- Requires very few plaintext–ciphertext pairs (~ 1).
- No attack against "industrial strength" cipher faster than other techniques known.
- Algorithms: Bucherberger algorithm [7], F_4 [9], F_5 [10], Raddum-Semaev [11], SAT-solvers [1], XL family [6]
- Often statistical components: SAT-solvers, key bit guessing, AES inversion equations $xy + 1$ [5].

Multiple $P - C$ Pairs [8] I

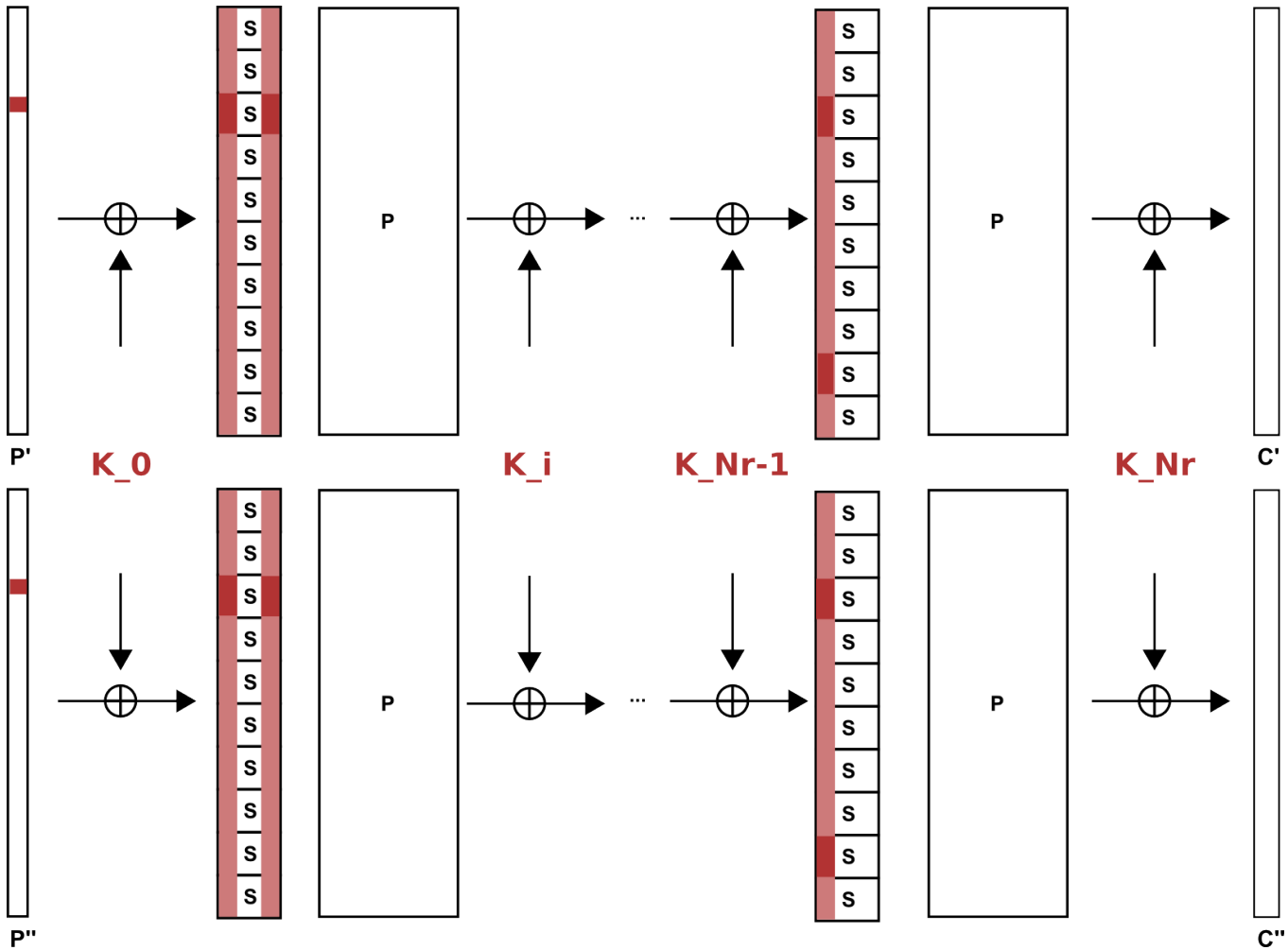
- Given two equation systems F' and F'' for two plaintext-ciphertext pairs (P', C') and (P'', C'') under same encryption key K.
- We can combine these equation systems to form a system $F = F' \cup F''$.
- While F' and F'' do not share most of the state variables X', X'', Y', Y'' but they share the key K and key schedule variables K_i.
- Thus by considering two plaintext–ciphertext pairs the cryptanalyst gathers twice as many equations, involving however many new variables.

Multiple $P - C$ Pairs [8] II



Outline

Attack-A I



Attack-A II

- Each one-round difference gives rise to equations relating the input and output pairs for active S-Boxes.
- We have that the expressions

$$X'_{j,k} + X''_{j,k} = \Delta X_{j,k} \rightarrow \Delta Y_{j,k} = Y'_{j,k} + Y''_{j,k},$$

  where $\Delta X_{j,k}, \Delta Y_{j,k}$ are known values predicted by the characteristic, are valid with some non-negligible probability p_j.
- For non-active S-Boxes we have the relations

$$X'_{j,k} + X''_{j,k} = 0 = Y'_{j,k} + Y''_{j,k}$$

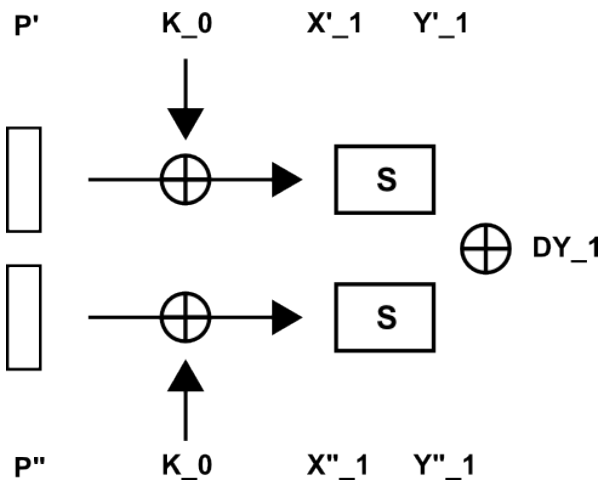
  also valid with a non-negligible probability.

These are $2n$ linear equations per round we can add to our equation system F. The resulting system $\overline{F}$ is expected to be easier to solve but we need to solve $1/p$ such systems.

Attack-B I

Restrict the first round bits to an active S-Box and assume we have a right pair. Also let the S-Box be representable by the vectorial Boolean function

$$S(X_i) = \begin{matrix} f_0(X_{i,0}, \ldots, X_{i,n-1}) \\ \ldots \\ f_{n-1}(X_{i,0}, \ldots, X_{i,n-1}) \end{matrix}.$$



If $P' - C'$ and $P'' - C''$ is a right pair, we have

- $S(P' \oplus K_0) = S(X_1') = Y_1'$
- $S(P'' \oplus K_0) = S(X_1'') = Y_1''$
- $Y_1' \oplus Y_1'' = \Delta Y_1$

$\rightarrow S(P_1' \oplus K_0) \oplus S(P_1'' \oplus K_0) = \Delta Y_1$

Attack-B II

We can use this small equation system F_s to recover bits of information about the subkey. Specifically:

Lemma

Given a differential characteristic Δ with a first round active S-Box with a difference that is true with probability 2^{-b}, then by considering F_s we can recover b bits of information about the key from this S-Box.

In the case of PRESENT we can learn 4-bit of information per characteristic Δ.

Attack-B III

Experimental Observation

For some ciphers **Attack-A** can be used to distinguish **right pairs** and thus enables this attack.

Attack-B proceeds by measuring the time t it maximally takes to find that the system is inconsistent and assume we have a right pair if this time t elapsed without a contradiction.

Attack-B IV

N_r	K_s	r	p	SINGULAR	POLYBORI
4	80	4	2^{-16}	11.92-12.16	0.72 - 0.81
4	80	3	2^{-12}	106.55-118.15	6.18 - 7.10
4	80	2	2^{-8}	119.24-128.49	5.94 - 13.30
4	80	1	2^{-4}	137.84-144.37	11.83- **33.47**
16	80	14	2^{-62}	N/A	43.42-64.11
16	128	14	2^{-62}	N/A	45.59-65.03
16	80	13	2^{-58}	N/A	80.35-**262.73**
16	128	13	2^{-58}	N/A	81.06-320.53
16	80	12	2^{-52}	N/A	>4 hours
17	80	14	2^{-62}	12,317.49-13,201.99	55.51 - 221.77
17	128	14	2^{-62}	12,031.97-13,631.52	94.19 - 172.46
17	80	13	2^{-58}	N/A	>4 hours

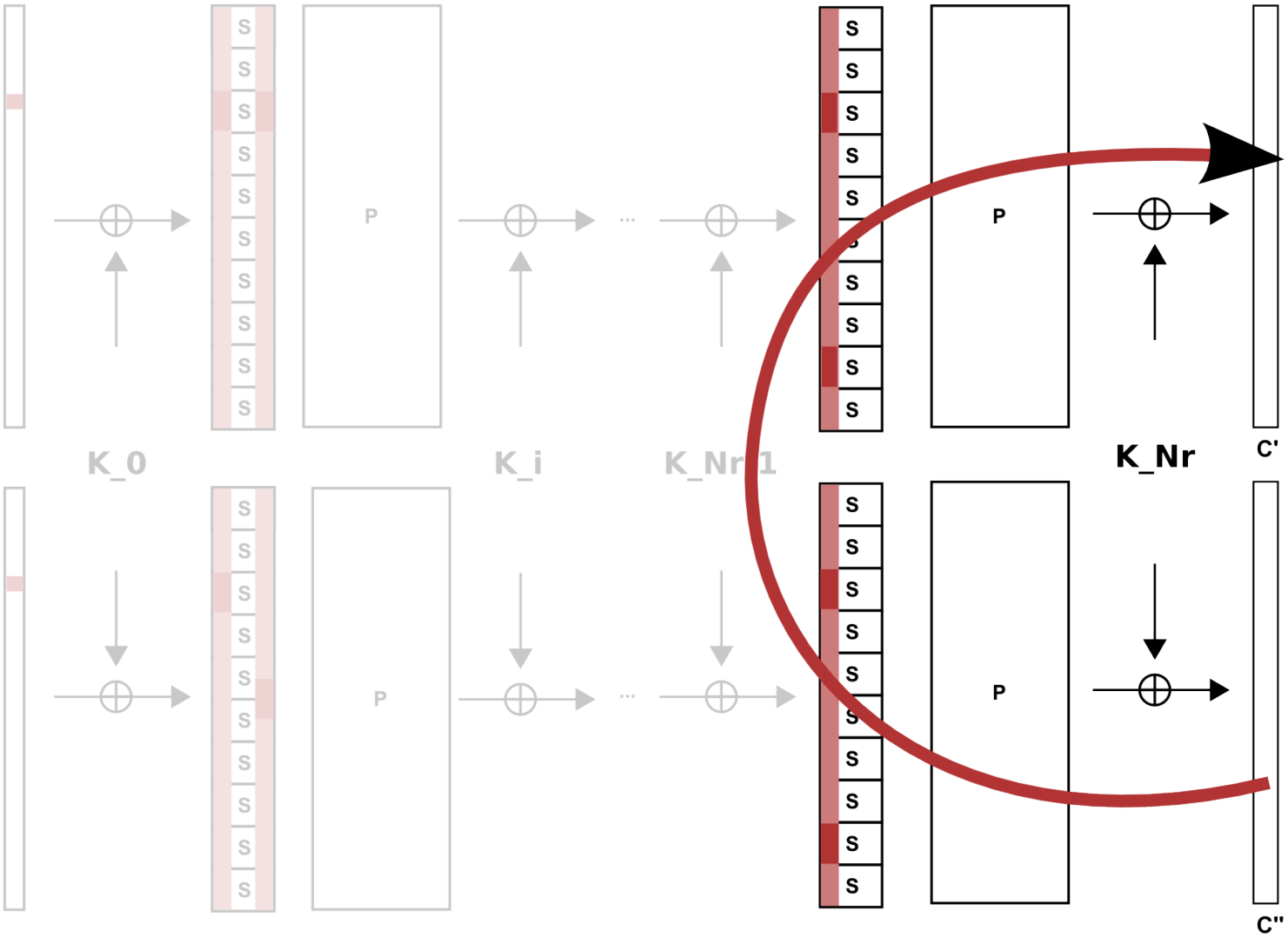
Table: Times in seconds for **Attack-B**

Times obtained on William Stein's `sage.math.washington.edu` computer purchased under NSF Grant No. 0555776.

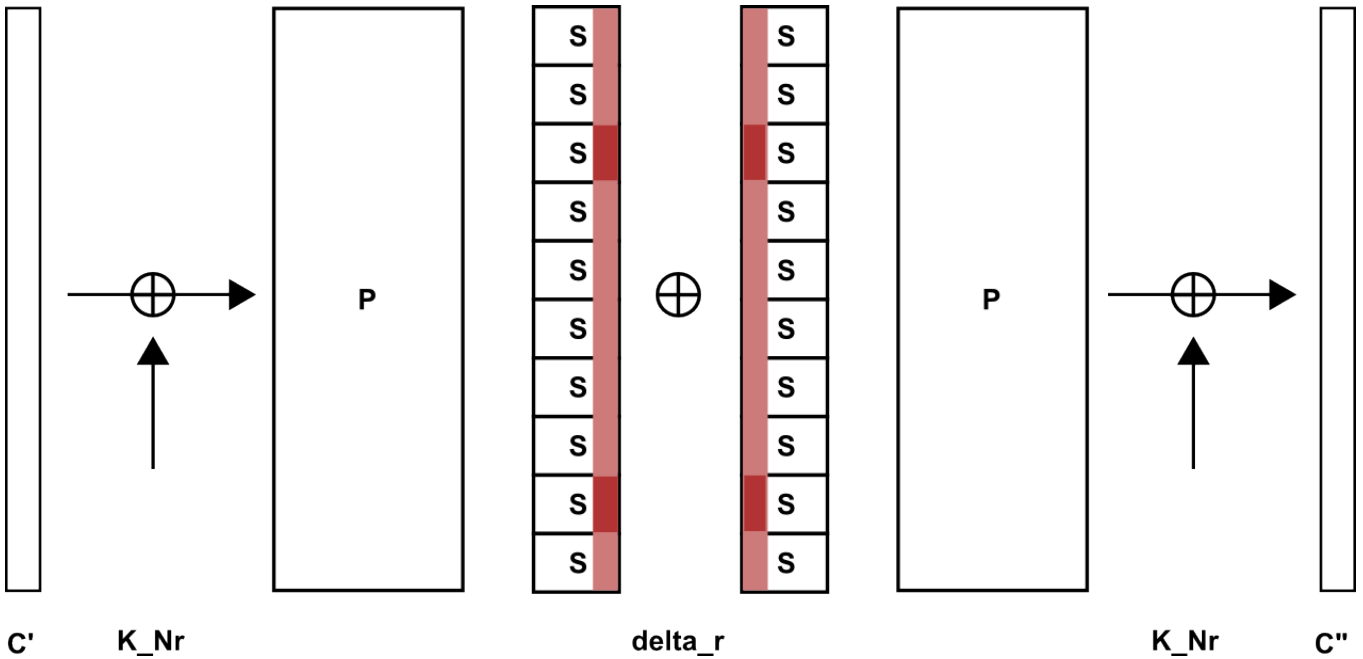
Why?

$$\frac{262.73\ s}{33.47\ s} \approx 7.85$$

Attack-C I



Attack-C II



The algebraic computation is essentially equivalent to solving a related cipher of $2(N_r - r)$ rounds (from C' to C'' via the predicted difference δ_r) with a symmetric key schedule, using an algebraic meet-in-the-middle attack.

Attack-C III

In a Nutshell

Attack-C is an algebraic filter.

Attack-C IV

N_r	r	p	#trials	K_s	t for PolyBoRi	K_s	t for PolyBoRi
4	4	2^{-16}	50	80	0.05 – 0.06	128	N/A
4	3	2^{-12}	50	80	0.88 – 1.00	128	N/A
4	2	2^{-8}	50	80	2.16 – 5.07	128	N/A
4	1	2^{-4}	50	80	8.10 – 18.30	128	N/A
16	14	2^{-62}	100	80	2.38 – 5.99	128	2.38 – 5.15
16	13	2^{-58}	100	80	8.69 – 19.36	128	9.58 – 18.64
17	14	2^{-62}	100	80	9.03 – 16.93	128	8.36 – 17.53

Table: Times in seconds for **Attack-C**

Outline

Present-80-16 I

4 bits:

- **Filter:** $(1 \pm \epsilon) \cdot 2^{62}$ ciphertext checks
- **Algebraic Filter:** $(1 \pm \epsilon) \cdot 2^{11.93} \cdot 6 \cdot 1.8 \cdot 10^9 \approx 2^{46}$ cpu cycles

Full Key Recovery:

- **Characteristics:** 6 characteristics from [13]
- **Filter:** $6 \cdot (1 \pm \epsilon) \cdot 2^{62}$ ciphertext checks
- **Algebraic Filter:** $6 \cdot (1 \pm \epsilon) \cdot 2^{46}$ cpu cycles
- **Guess:** $80 - 18 = 62$ bits

Present-128-18

Consider the input difference for round 15 and iterate over all possible output differences. For the exampel difference we have 36 possible output differences for round 15.

Full Key Recovery:

- **Algebraic Filter:** $6 \cdot (1 \pm \epsilon) \cdot 36 \cdot 2^{62} \cdot 18 \cdot 1.8 \cdot 10^9 \approx 2^{102}$ cpu cycles
- **Guessing:** $128 - 18 = 110$ bits

Experimental Results Summary

Attack	N_r	K_s	r	#pairs	time	#bits	$K_s - \#bits$
Wang	16	80	14	2^{63}	2^{65} MA	57	23
Attack-C	16	80	14	2^{62}	2^{62} MA	4	76
Attack-C	16	80	14	$6 \cdot 2^{62}$	2^{62} encr.	18	62
Attack-C	18	128	14	2^{62}	2^{102} cycles	4	124
Attack-C	18	128	14	$6 \cdot 2^{62}$	2^{110} encr.	128	110

Outline

Discussion

Properties:

- One right pair is sufficient to learn some information about the key.
- No requirement for candidate key counter.
- Silimar to DC attack on full DES [3] but **in theory** applicable to any block cipher.

Possible improvements are

- better algebraic representations,
- better algorithms (e.g. SAT-solvers) and
- better exploitation of right pair.

PRESENT-128-(18+i)?

Speculation

It might be possible to find contradictions using **Attack-C** in $\ll 2^{128-62} = 2^{66}$ cpu cycles for PRESENT-128-20 "a situation without precedent" [4].

Conclusion

- We presented a new approach which uses algebraic techniques in differential cryptanalysis.
- Specifically, we show how to invest more time in the last rounds not covered by a differential.
- To illustrate the viability of the attack we improved the best known attack against PRESENT-128 by two rounds using the same characteristics.

Note

This attack has no implication for the security of PRESENT!

Thank you!

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