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Cold Boot Key Recovery using Polynomial System Solving with Noise

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Outline

Outline

Coldboot Attacks I

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- In [3] a method is described for extracting cryptographic key material from DRAM.
- DRAM may retain large part of its content for several seconds after removing its power.
- Furthermore, time can potentially be increased by using cooling techniques.
- In the case of the AES and DES simple algorithms are also proposed in [3] to recover the key from the observed set of round subkeys in memory, which are however subject to errors (due to memory bits decay).

Coldboot Attacks II

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We are given

1. an efficiently computable function $\mathcal{KS} : \mathbb{F}_2^n \to \mathbb{F}_2^N$ with $N > n$,
2. two real numbers $0 \leq \delta_0, \delta_1 \leq 1$ and
3. some efficiently computable function $\mathcal{E}(k) \to \{True, False\}$.

Let $(K_0, \dots, K_{N-1}) = \mathcal{KS}(k)$. Compute $(K'_0, K'_1, \dots, K'_{N-1})$ with:

$$\begin{array}{rclcrcl} Pr[K'_i = 0 \mid K_i = 0] & = & 1 - \delta_1, & & Pr[K'_i = 1 \mid K_i = 0] & = & \delta_1, \\ Pr[K'_i = 1 \mid K_i = 1] & = & 1 - \delta_0, & & Pr[K'_i = 0 \mid K_i = 1] & = & \delta_0. \end{array}$$

$K'_i = 0$ is correct with probability $Pr[K_i = 0 \mid K'_i = 0] = \frac{(1-\delta_1)}{(1-\delta_1+\delta_0)} = \Delta_0$. Likewise for $K'_i = 1$.

The task is to recover k such that $\mathcal{E}(k)$ returns *True* or a noise-free K.

Coldboot Attacks III

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Results in [3]:

Cipher	δ_0	δ_1	Success	Time
DES	0.10	0.001	100%	–
DES	0.50	0.001	98%	–
AES	0.15	0.001	100%	1s
AES	0.30	0.001	100%	30s

Can we do better and can we recover keys for more complicated key schedules such as Serpent?

Outline

PoSSo

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We define polynomial system solving (**PoSSo**) as the problem of finding a solution to a system of polynomial equations over some field.

Definition (PoSSo)

Consider the set $F = \{f_0, \dots, f_{m-1}\}$ where each $f_i \in \mathbb{F}[x_0, \dots, x_{n-1}]$.

A solution to F is any point $x \in \mathbb{F}^n$ such that

$$\forall f_i \in F : f_i(x) = 0.$$

Note, that we restrict ourselves to solutions in the base field here.

Max-PoSSo I

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We can define a family of **Max-PoSSo** problems, analogous to the well known Max-SAT family of problems.

`http://en.wikipedia.org/wiki/MAX-SAT`

In fact, we can reduce Max-PoSSo to Max-SAT.

Max-PoSSo II

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Definition (Max-PoSSo)

Find a point $x \in \mathbb{F}^n$ which satisfies the **maximum number** of polynomials in $F = \{f_0, \dots, f_{m-1}\} \subset \mathbb{F}[x_0, \dots, x_{n-1}]$.

Max-PoSSo III

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Definition (Partial Weighted Max-PoSSo)

Find a point $x \in \mathbb{F}^n$ such that for **two sets of polynomials** $\mathcal{H}$ and $\mathcal{S} \subset \mathbb{F}[x_0, \ldots, x_{n-1}]$

- $\forall f \in \mathcal{H} : f(x) = 0$ and
- $\sum_{f \in \mathcal{S}} \mathcal{C}(f, x)$ is minimized

where $\mathcal{C} : f \in \mathcal{S}, x \in \mathbb{F}^n \to \mathbb{R}_{\geq 0}$ is a **cost function** which

- returns 0 if $f(x) = 0$ and
- some value > 0 if $f(x) \neq 0$.

Coldboot as P. W. Max-PoSSo

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- Let $F_{\mathcal{K}}$ be an equation system corresponding to $\mathcal{K}$.
- Assume that for each noisy output bit K_i there is some $f_i \in F_{\mathcal{K}}$ of the form $g_i + K_i$ where g_i is some polynomial.
- Assume that these are the only polynomials involving output bits.
- Denote the set of these polynomials $\mathcal{S}$.
- Denote the set of all remaining polynomials $\in F_{\mathcal{K}}$ as $\mathcal{H}$.
- Define the cost function $\mathcal{C}$ as a function which returns

$$\begin{array}{cr} \frac{1}{1-\Delta_0} & \text{for } K_i' = 0, f(x) \neq 0, \\ \frac{1}{1-\Delta_1} & \text{for } K_i' = 1, f(x) \neq 0, \\ 0 & \text{otherwise.} \end{array}$$

- Express $\mathcal{E}$ as a polynomial system which is satisfiable for k only and add these polynomials to $\mathcal{H}$.

Outline

1 Coldboot Attacks

2 Polynomial System Solving with Noise

3 Mixed Integer Programming

4 Application

Mixed Integer Programming I

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Integer optimization deals with the problem of minimising (or maximising) a function in several variables subject to linear equality and inequality constraints and integrality restrictions on some or all of the variables.

We minimise (or maximise) a linear function $c^T x$ subject to linear equality and inequality constraints given by some matrix A and a vector b as $Ax \leq b$.

We have that some variables are restricted to integer values while other variables are real-valued.

Mixed Integer Programming II

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The set S of all $x \in \mathbb{Z}^k \times \mathbb{R}^l$ which satisfies the linear constraints $Ax \leq b$

$$S = \{x \in \mathbb{Z}_k \times \mathbb{R}_l, Ax \leq b\}$$

is called the feasible set.

If $S = \varnothing$ the problem is infeasible. Any $x \in S$ which minimises (or maximises) $c^T x$ is an optimal solution.

PoSSo as MIP I

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Consider some $f \in \mathbb{F}_2[x_0, \dots, x_{n-1}]$ and let $\mathcal{Z}$ a function that takes a polynomial over $\mathbb{F}_2$ lifts it to the integers. Analogous for elements in $\mathbb{F}_2$.

1. Restrict all x_i to binary values.
2. Evaluate $\mathcal{Z}(f)$ on all $\{\mathcal{Z}(x) \mid x \in \mathbb{F}_2^n, f(x) = 0\}$.
3. Let ℓ be the minimum value and u the maximum value.
4. Introduce some integer variable $\frac{\ell}{2} \leq m \leq \frac{u}{2}$.
5. Replace each monomial in $f - 2m$ by a new linearised variable, call the result g and add the linear constraint $g = 0$.
6. For each monomial $t = \prod_{i=1}^{N} x_i$
   - add a constraint $x_i \geq t$ and
   - add a constraint $0 \leq \sum_{i=1}^{N} x_i - t \leq N - 1$.

This is the **Integer Adapted Standard Conversion** [1].

Partial Weighted Max-PoSSo as MIP

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- Convert each $f \in \mathcal{H}$ to linear constraints as before.
- For each $f_i \in \mathcal{S}$ add a new binary slack variable e_i to f_i and convert the resulting polynomial as before.
- The objective function we minimise is $\sum c_i e_i$ where c_i is the value of $\mathcal{C}(f, x)$ for some x such that $f(x) \neq 0$.

Any optimal solution $x \in S$ will be an optimal solution to the Weighted Partial Max-PoSSo problem.

Coldboot as MIP

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Coldboot → Partial Weighted Max-PoSSo → MIP

This approach is essentially the non-linear generalisation of decoding random linear codes with linear programming [2].

Outline

Simplifications

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- We do not model $\mathcal{E}$ since its representation is often too costly; consequently we have no guarantee that the optimal k returned is indeed the k we are looking for.
- We do not include all equations available to us but restrict our attention to a subset (e.g. one or two rounds).
- We may use an "aggressive" modelling strategy where we assume $\delta_1 = 0$ which allows us to promote some polynomials from $\mathcal{S}$ to $\mathcal{H}$.

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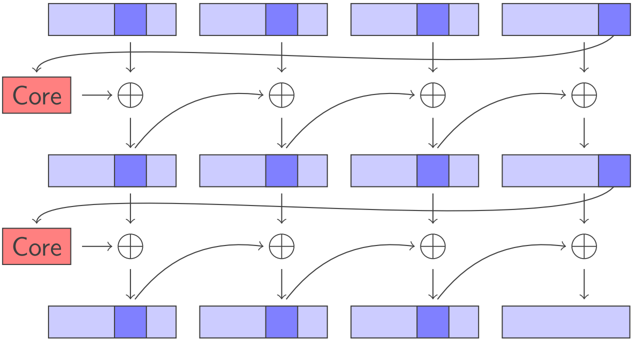
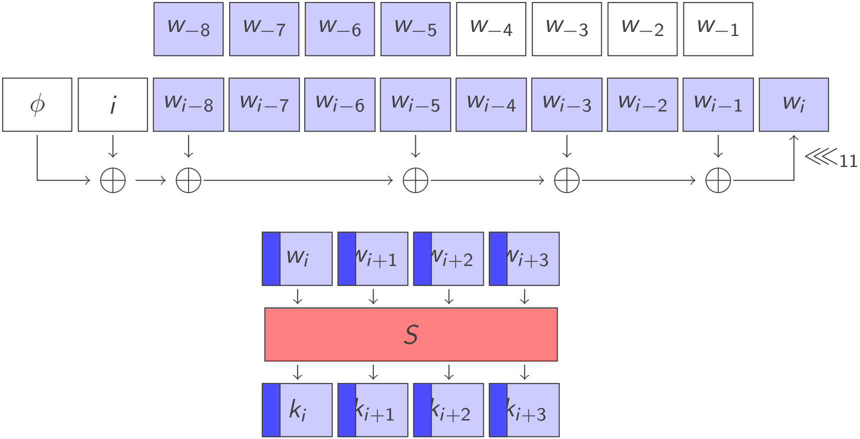
AES I



AES II

		Gurobi `http://www.gurobi.com`				
N	δ_0	a	#cores	cutoff t	r	max t
3	0.15	–	24	∞	100%	17956.4s
3	0.15	–	2	240.0s	25%	240.0s
3	0.30	+	24	3600.0s	25%	3600.0s
3	0.35	+	24	28800.0s	30%	28800.0s
		SCIP `http://scip.zib.de`				
3	0.15	+	1	3600.0s	65%	1209.0s
4	0.30	+	1	7200.0s	47%	7200.0s
4	0.35	+	1	10800.0s	45%	10800.0s
4	0.40	+	1	14400.0s	52%	14400.0s
5	0.40	+	1	14400.0s	45%	14400.0s

Serpent I



w_{-8} w_{-7} w_{-6} w_{-5} w_{-4} w_{-3} w_{-2} w_{-1}
ϕ i w_{i-8} w_{i-7} w_{i-6} w_{i-5} w_{i-4} w_{i-3} w_{i-2} w_{i-1} w_i
$\lll 11$
w_i w_{i+1} w_{i+2} w_{i+3}
S
k_i k_{i+1} k_{i+2} k_{i+3}

Serpent II

		Gurobi http://www.gurobi.com				
N	δ_0	a	#cores	cutoff t	r	Max t
8	0.05	–	2	60.0s	50%	16.22s
12	0.05	–	2	60.0s	85%	60.00s
8	0.15	–	24	600.0s	20%	103.17s
12	0.15	–	24	600.0s	55%	600.00s
12	0.30	+	24	7200.0s	20%	7200.00s
		SCIP http://scip.zib.de				
12	0.15	+	1	600.0s	32%	597.37s
16	0.15	+	1	3600.0s	48%	369.55s
20	0.15	+	1	3600.0s	29%	689.18s
32	0.15	+	1	3600.0s	21%	1105.58s
16	0.30	+	1	3600.0s	55%	3600.00s
20	0.30	+	1	7200.0s	57%	7200.00s

Serpent III

Ad-hoc approach:

- We wish to recover a 128-bit key, so we need to consider at least 128-bit of output.
- On average the noise free output should have 64 bits set to zero.
- In order to consider an error rate up to δ_0, we have to consider

$$\sum_{i=0}^{\lceil \delta_0 \cdot 64 \rceil} \binom{64 + \lceil \delta_0 \cdot 64 \rceil}{i}$$

  candidates and test them.
- If $\delta_0 = 0.15$ we have $\approx 2^{36.87}$.
- If $\delta_0 = 0.30$ we have $\approx 2^{62}$.

Thank you!

Literature I

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