Experimental Robotics (CS225A) Handout #06

(Spring 2012-13)

Minimum time required in tracking cubic splines

We are given θ_0, θ_f, and the fact that $\dot{\theta}_0 = \dot{\theta}_f = 0$. We want to find a cubic segment that will interpolate these states with a guarantee that the velocity and acceleration will stay below some specified values:

$$|\dot{\theta}(t)| \le \dot{\theta}_{\max}, \quad |\ddot{\theta}(t)| \le \ddot{\theta}_{\max}$$

If we solve $\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ for $\theta(0) = \theta_0, \theta(t_f) = \theta_f, \dot{\theta}(0) = 0$, and $\dot{\theta}(t_f) = 0$, we get:

$$a_0 = \theta_0, \quad a_1 = 0, \quad a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0), \quad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

The only remaining free variable t_f. So we will find a t_f that satisfies the $\dot{\theta}_{\max}$ and $\ddot{\theta}_{\max}$ constraints, and then find the coefficients of the spline from the above equations.

In order to satisfy the constraints, we need to examine the extrema of $\dot{\theta}(t)$, and $\ddot{\theta}(t)$. In other words

$$|\dot{\theta}(t)| \le |\dot{\theta}_{ex}| \le \dot{\theta}_{\max} \text{ and } |\ddot{\theta}(t)| \le |\ddot{\theta}_{ex}| \le \ddot{\theta}_{\max}$$

The velocity is maximized at $t_f/2$, since $\dot{\theta}(t)$ is a parabola with zeroes at 0 and t_f:

$$\begin{aligned} & \dot{\theta}_{ex} = |\dot{\theta}(t_f/2)| \\ \Rightarrow \quad & \left| a_1 + 2a_2 \frac{t_f}{2} + 3a_3 \frac{t_f^2}{4} \right| \le \dot{\theta}_{\max} \\ \Rightarrow \quad & \left| \frac{3}{t_f}(\theta_f - \theta_0) - \frac{3}{2t_f}(\theta_f - \theta_0) \right| \le \dot{\theta}_{\max} \\ \Rightarrow \quad & \left| \frac{3}{2t_f}(\theta_f - \theta_0) \right| \le \dot{\theta}_{\max} \end{aligned}$$

So the velocity constraint is satisfied if:

$$t_f \ge \frac{3}{2\dot{\theta}_{\max}} |\theta_f - \theta_0|$$

Similarly, to satisfy the acceleration contstraint, note that $\ddot{\theta}(t)$ is a linear function with extrema at $t = 0$ and $t = t_f$:

$$\begin{aligned} & |\ddot{\theta}_{ex}| = |\ddot{\theta}(0)| \\ \Rightarrow \quad & |2a_2 + (6a_3)(0)| \le \ddot{\theta}_{\max} \\ \Rightarrow \quad & \left| \frac{6}{t_f^2}(\theta_f - \theta_0) \right| \le \ddot{\theta}_{\max} \end{aligned}$$

So the acceleration constraint is satisfied if

$$t_f \ge \sqrt{\frac{6}{\ddot{\theta}_{\max}} |\theta_f - \theta_0|}$$

Putting these two conditions together, the constraint on t_f is

$$t_f \ge \max\left(\frac{3}{2\dot{\theta}_{\max}} |\theta_f - \theta_0|, \sqrt{\frac{6}{\ddot{\theta}_{\max}} |\theta_f - \theta_0|} \right)$$